A Rolling Recruitment Process under Applicant Stochastic Departures

Chenyin Gong* Qing Li†

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Abstract

We study a rolling recruitment process in which applicants leave the system stochastically. Applicants arrive randomly over time, and each applicant is available for a random amount of time after they arrive. In each period, the recruiter must decide whether to *stop* or to *wait*. If they stop, they need to determine how many offers to make and whom to make offers to, and the applicants who do not receive an offer will leave the system. If the recruiter waits, more applicants will be available in the next period, while some who arrived earlier will leave. We model the process as a large-scale optimal stopping problem and show how the applicant qualifications, measured by scores, affect the recruiter's optimal policy. We find that the optimal stopping rule for each applicant's score is a two-threshold policy. If the score exceeds the higher threshold, the recruiter stops and makes an offer to the applicant and possibly to others. If the score falls below the lower threshold, the recruiter also stops but makes no offer to the applicant. If the score is in between the two thresholds, the recruiter waits. We further explore the impact on an applicant's likelihood of receiving an offer if their competitors become more qualified. When the score of another applicant increases, the recruiter may change from making an offer to an applicant to waiting or to instead making an offer to the other applicant whose score has increased. In other words, an applicant may be disadvantaged if they face stronger competitors, which is expected. However, an applicant may also benefit from having stronger competitors. When the score of another applicant increases, the recruiter may change from not making an offer to an applicant to making an offer to both applicants. Overall, we provide valuable insights into the role of applicant qualifications in stopping decisions, propose methods for computing the optimal policy, and quantify the benefits of endogenously determining the stopping rule.

*School of Business and Management, Hong Kong University of Science and Technology, Clear Water Bay, Kowloon, Hong Kong SAR. email: cgongad@connect.ust.hk

†ibid., email: imqli@ust.hk

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1 Introduction

A sequential or rolling recruitment process is one in which a recruiter needs to fill a number of job vacancies over a recruitment season and applicants can submit their applications at any time before the end of the season. The recruiter assesses the qualifications of the applicants as they arrive and may extend offers at any period during the season. The rolling process can spread out the workload for the recruiter, provide time flexibility to applicants, and enhance the applicants' experiences (Du et al. 2024). In managing a rolling process, one practical and challenging question that the recruiter must answer is when to make offers to applicants and how many offers to make. To see the trade-off facing the recruiter, let us consider two extreme policies regarding the timing of making offers. At one extreme, the recruiter can make offers to applicants as soon as they arrive in a period. Under this policy, the recruiter may risk making offers to applicants who arrive earlier but who are less qualified than those who arrive later. At the other extreme, the recruiter can wait until the end of the season, at which point they can rank all of the applicants and determine whom to make offers to. However, some of the applicants who arrived earlier may have already accepted offers elsewhere and will thus no longer be available. In determining when to make offers and how many offers to make, the recruiter is confronted with three types of uncertainty at any given point: how many applicants will arrive in the future, how qualified they are, and how long those who have submitted applications will remain available.

The question above is at the very heart of managing a rolling recruitment process. Without a good answer, the full potential of a rolling process cannot be realized. Because of the uncertainty involved, this question is complex, and without an analytical tool, it is impossible to understand the trade-off, let alone provide a good answer. In this paper, we model the process as a large-scale, discrete-time optimal stopping problem. A random number of applicants arrive in each period, and the recruiter assesses their qualifications, which are measured by scores, as soon as they arrive. The recruiter makes two decisions in each period. First, they must decide whether to stop or to wait. If they stop, they need to determine how many offers to make and to whom. The applicants who do not receive an offer will leave the system and cannot be recalled. If the recruiter waits, more applicants will be available in the next period, while some who arrived earlier will leave. The recruiter's objective is to maximize the total reward, which is measured by the total scores of the hired applicants minus the penalty cost associated with the potential mismatch between the total number of vacancies and the number of applicants hired in the end. Intuitively, the recruiter's decisions depend on the qualifications of the applicants who have arrived but have not departed. The focus of our study is on the impact of the applicants' qualifications on the recruiter's optimal policy and total reward.

For each applicant's score, there are two thresholds. If the score exceeds the higher threshold, the recruiter stops and makes an offer to the applicant, and possibly to others. If the score falls below the lower threshold, the recruiter also stops but does not make an offer to the applicant. If the score is between the two thresholds, the recruiter waits. Somewhat surprisingly, the recruiter may change from stopping to waiting when an applicant's score increases. When the recruiter waits, they risk losing the applicants with high scores and hence being left with those with low scores. When an applicant's score increases from very low to intermediate, the recruiter's risk of being left with applicants with very low scores decreases, which provides a stronger incentive to wait.

What is the impact on an applicant's likelihood of receiving an offer if their competitors become more qualified? When the score of another applicant increases, the recruiter may change from making an offer to one applicant to making an offer to another applicant whose score has increased. In other words, an applicant may be disadvantaged if they face stronger competitors, which is expected. However, when the score of another applicant increases, the recruiter may also change from waiting to making offers to both applicants, or change from stopping and making offers to neither applicant to stopping and making offers to both. In this case, an applicant benefits from having stronger competitors. In the former case, this "free riding" phenomenon occurs because once the recruiter stops, all applicants leave the system, and it will take time for the applicant pool to thicken again. The recruiter therefore lowers the cutoff. In the latter case, it occurs because hiring both, as opposed to hiring only one, moves the total number of hired applicants closer to the target, and this may allow the recruiter to be more patient in waiting for qualified applicants in later periods. Finally, we show that the recruiter benefits from having a pool of applicants with more dispersed scores in the system or applicants with stochastically more variable scores arriving in the future.

Although our work is motivated by rolling recruitment processes, our model and results are applicable to practices such as dynamic auctions, investment, and selling assets. For example, on online auction platforms such as eBay, a seller conducts auctions to sell a product with multiple units, and buyers with different reservation values bid on the product during each bid episode (Vulcano et al. 2002). In sequential investment, an investor with access to a limited pool of capital decides the investment amount for each sequentially revealed opportunity (e.g., advertising), and the return depends on the investment amount and the quality of the opportunity (Prastacos 1983). One thing in common among all of these practices is that limited resources must be allocated to opportunities that arrive randomly over time. The opportunities are available for consideration only for a limited and random amount of time; this has been called *stochastic departures* in the literature (Karni and Schwartz 1977, Kesselheim et al. 2024).

In a rolling recruitment process, the recruiter benefits from exercising the option to waiting; that is, they determine endogenously whether they should stop or wait in each period depending on the number and qualifications of the applicants already in the system. The benefits of waiting in operations are well documented. In online retailing, for example, when retailers such as Amazon receive an order, there is a time delay for picking and packing the order to smooth the workload of its warehouses or to prioritize shipping for urgent orders (Xu et al. 2009). In emergency operations, after treating patients, emergency department physicians determine whether to admit their patients for inpatient care or discharge them. Physicians may postpone requesting an inpatient bed to batch more patients, which is a practical way to improve their productivity (Feizi et al. 2023). In all of these practices, it is important to make waiting decisions judiciously to prevent excessive departures, which means order cancellations in retailing and patient elopement in health care (i.e., a patient leaves the hospital against medical advice, which may pose an imminent threat to the patient's health or safety). We numerically evaluate the value of waiting. Our numerical studies show that the value can exceed 10% and that it is high when highly qualified applicants arrive infrequently, the total number of vacancies is not too high, the arrival rate is low, or the departure probability is low.

The rest of the paper is organized as follows. In Section 2, we provide a discussion of the related literature. We formulate the model in Section 3, present the structural properties of the optimal policy in Section 4, and investigate the impact of score dispersion in Section 5. In Section 6, we propose a threshold-based heuristic inspired by our theoretical results and use it to numerically examine the value of endogenously determining the stopping rule. In Section 7, we extend our analysis to a more general setting that involves waiting lists. We conclude the paper in Section 8.

2 Related Literature

Sequential recruitment has traditionally been studied under the assumption that applicants arrive one at a time and the recruiter must reject or accept them on the spot (e.g., Arlotto and Gurvich

2019, Vera and Banerjee 2021, Arnosti and Ma 2023). In Li and Yu (2021), however, applicants are assessed and offers are made periodically in batches. When applicants are processed in batches, the recruiter must rank the applicants who arrive in that period, and the reject/accept decision for each applicant may depend on the qualifications of all of the applicants. Similarly, in Du et al. (2024), applicants arrive and are assessed in batches, but applicants who receive offers may reject them. The incorporation of random yields complicates the model. Therefore, instead of finding optimal policies, they focus on deriving simple and asymptotically optimal heuristics and test them in a case study. Our study differs from the literature in two significant ways. First, in the literature, the recruiter must stop to assess applicants and make offers either as soon as applicants arrive or at a fixed sequence of periods, while our model allows the recruiter to endogenously determine when to stop, depending on the number and qualifications of the applicants in the system. Second, prior studies do not consider stochastic departures, which is a major element in our model.

Sequential recruitment is an example of more general sequential assignment problems, where a fixed amount of a resource is assigned to opportunities that arrive randomly over time to maximize the total expected reward. Many applications fall under the category of sequential assignment problems. For example, Prastacos (1983) examines optimal investment strategies for an investor who faces a sequence of opportunities with random qualities, where the investment return depends on the amount of capital allocated and the quality of the opportunity. Ahn et al. (2021) formulate an asset-selling problem involving debt obligations, where the seller decides on the selling portion in each period after observing the selling price. In Zhang and Swaminathan (2020), a farmer must allocate seed amount in each period under rainfall uncertainty. In Vulcano et al. (2002), a seller sells a fixed amount of inventory through dynamic auctions and, as in Li and Yu (2021), bids are evaluated in a fixed, predetermined sequence of intervals. Xie et al. (2023) focus on the benefits of waiting in a class of sequential assignment problems. To our best knowledge, none of the above studies allows the lengths of the intervals to be chosen endogenously or considers stochastic departures.

Our work is also intimately connected to the extensive literature on the secretary problem.[1] In the classic formulation of the secretary problem, n applicants are presented sequentially to a

[1] Here we define the secretary problem narrowly as the one in which the objective is to maximize the probability of choosing the best applicants, and the payoff depends on the observations of their relative ranks and not on their actual values. Similar definitions can be found, for example, in Ferguson (1989). However, some scholars have defined the secretary problem more broadly (e.g., Arnosti and Ma 2023) and viewed it as an example of sequential assignment problems.

recruiter, who accepts or rejects each applicant based on their relative ranks. The recruiter wishes to maximize the probability of choosing the best applicant. Earliest work on the secretary problem includes Gardner (1960), Lindley (1961), and Gilbert and Mosteller (1966), to name a few. For reviews of studies on the secretary problem and its variations, readers can refer to Freeman (1983), Ferguson (1989), and DeGroot (2004). Some extensions of the classic secretary problem are more relevant to our work. For example, Goldys (1978) and Ho and Krishnan (2015) address a scenario in which the decision for each job vacancy can be delayed until after a fixed number of additional applicants have been seen, which is sometimes called a "sliding window." Kesselheim et al. (2024) consider applicant stochastic departures, which seem to be more realistic than a sliding window. Obviously, because of differing objectives, the tools that we use for analysis and the optimal policy for our problem are different from those for the secretary problem.

One of the key features in our model is stochastic departures. There is considerable interest in stochastic departures in other contexts. For example, a large strand of the queuing theory literature deals with stochastic scheduling in the presence of customer abandonment, where customers in a service system may become impatient and abandon the system without receiving service. For a detailed review, refer to Gans et al. (2003). The trade-off between waiting and market thickness in the presence of stochastic departures is also extensively studied in the matching theory literature. See Akbarpour et al. (2020) and Ashlagi et al. (2023) for recent reviews.

In summary, although the body of related literature is voluminous, our work is the first to study a rolling recruitment process in which the recruiter's objective is to maximize the total reward, the recruiter can determine the timing of making offers endogenously, and applicants leave the system stochastically. In addition, our study centers on the structural properties of the recruiter's optimal policy and total reward with respect to applicant qualifications.

3 The Model

In the following, we use bold letters to represent vectors and $\mathbf{e}_i$ to denote the vector with 1 in the ith coordinate and 0 in all other coordinates. Let $\mathbf{s} = (s_1, s_2, \ldots, s_n)$. The dimensions of $\mathbf{e}_i$ and $\mathbf{s}$ should be clear from the context. Let $\mathbf{s}_{-j} = (s_1, \ldots, s_{j-1}, s_{j+1}, \ldots, s_n)$ and $(s_i)_{i \in I}$ be defined the same as $\mathbf{s}$ but only keeping the coordinates in $I \subset \{1, 2, \ldots, n\}$, e.g., $(s_i)_{i \in \{1,n\}} = (s_1, s_n)$. Denote the size of a finite set A by $|A|$ and the power set by 2^A. Let $x^+ = \max\{0, x\}$ and $x^- = \max\{0, -x\}$. Let $\nabla_{x_i} f(\mathbf{x})$ represent $(f(\mathbf{x} + \varepsilon \mathbf{e}_i) - f(\mathbf{x}))/\varepsilon$ for some small $\varepsilon > 0$.

A recruiter needs to fill d job vacancies in a recruitment season that is T periods long. A random number of applicants arrive in each period. Let the random variable N_t represent the number of applicants who are in the system in period t. Its realization is denoted by n_t. The value to the recruiter of hiring applicant i in period t is represented by an assessment score S_i^t, which is a positive and continuous random variable with a distribution function F. Let s_i^t be the realization of S_i^t. Each applicant arriving in a period departs the system in the next period with probability $p \in [0, 1]$. The random variable W_i^t represents whether applicant i who is in the system in period t departs in period $t+1$. It is equal to one if he or she departs in the next period and zero otherwise.[2] Let $\mathbf{W}^t = (W_1^t, W_2^t, \ldots, W_{N_t}^t)$, $\mathbf{S}^t = (S_1^t, S_2^t, \ldots, S_{N_t}^t)$, and $\mathbf{s}^t = (s_1^t, s_2^t, \ldots, s_{n_t}^t)$. The scores of applicants arriving across different periods are independent. Our main results continue to hold when the arrival process, the departure process, and the distribution of scores are nonstationary.

The timing of events in each period t is as follows. First, applicants arrive and the recruiter observes the total number of applicants hired so far, q_t, the number of applicants in the system, n_t, and their scores, $\mathbf{s}^t$. Second, the recruiter decides whether to stop or to wait. We define $a_t \in \{0, 1\}$ as the stopping decision. If the recruiter waits ($a_t = 0$), some applicants may depart at the end of the period; if the recruiter stops ($a_t = 1$), they rank the n_t applicants according to their scores and determine the number of offers, m_t, to make.[3] The applicants who do not receive an offer will leave the system and cannot be recalled later. At the end of the recruitment season, there is a penalty cost, measured by a convex function G, if the total number of applicants recruited, q_{T+1}, deviates from the hiring target d. It can be defined as, for example, $G(q_{T+1} - d) = u(q_{T+1} - d)^- + o(q_{T+1} - d)^+$, for some positive marginal underage and overage costs, u and o, respectively.[4]

Consistent with the literature on secretary problems, in the basic model, we assume that once the recruiter stops, the applicants who do not receive an offer will leave the system and there is no waiting list. In high-volume recruitment processes in which applicants are evaluated and offers are made batch by batch, if the market is very competitive, every time the recruiter stops to process a batch of applicants, the applicants whose scores are close but below the cutoff will take offers elsewhere. The applicants whose scores are far below the cutoff will not be considered even if they

[2]We model applicants' stochastic departures differently from Akbarpour et al. (2020), Ashlagi et al. (2023), and Kesselheim et al. (2024). In these papers, one can identify applicants who are about to depart, whereas in our model, the recruiter can only predict how likely each applicant is to depart at the end of a period.

[3]We can also add a fixed cost for each stop, but doing so will not affect our main results.

[4]In our model, meeting the target is not a hard constraint. In college admission, for example, recruiters often accept more or less applicants than the target, depending on the qualification of the applicants. If overage is not allowed, then we can set the overage cost sufficiently high. Underage must be allowed unless the total number of applicants during the entire season is guaranteed to be sufficiently large.

were placed on a waiting list (Du et al. 2024). In this environment, our assumption of waiting list is valid. We will discuss the implications of waiting lists in Section 7.

Let us define $\mathbf{S}_A^{t+1}(\mathbf{s}^t) = (s_i^t)_{i\in\left\{j\in\{1,2,\ldots,n_t\}:W_j^t=0\right\}}$, which represents the scores of the applicants who are in the system in period t and are still in the system in period $t+1$. Let $\mathbf{S}_B^{t+1}$ denote the scores of the applicants who arrive in period $t+1$. Clearly, $\mathbf{S}_A^{t+1}(\mathbf{s}^t)$ and $\mathbf{S}_B^{t+1}$ are independent. Given $\mathbf{s}^t$, the system dynamics with respect to the score information from period t to period $t+1$ are given by

$$\mathbf{S}^{t+1}(\mathbf{s}^t, a_t) = \begin{cases} (\mathbf{S}_A^{t+1}(\mathbf{s}^t), \mathbf{S}_B^{t+1}) & \text{if } a_t = 0, \\ \mathbf{S}_B^{t+1} & \text{if } a_t = 1. \end{cases}$$

In other words, if the recruiter waits, the score information in the next period consists of the scores of the applicants who arrived at or before period t and are still in the system in period $t+1$ and the scores of the applicants who arrive in period $t+1$; if the recruiter stops, the score information in the next period only includes the scores of the applicants who arrive in period $t+1$. The dynamic programming formulation is as follows:

$$V_t(q_t, \mathbf{s}^t) = \max\left\{\mathbb{E}V_{t+1}(q_t, \mathbf{S}^{t+1}(\mathbf{s}^t, 0)), \max_{1\le m_t\le n_t} J_t(q_t, m_t, \mathbf{s}^t)\right\}, \tag{1}$$

where

$$J_t(q_t, m_t, \mathbf{s}^t) = \sum_{i=1}^{m_t} s_{[i]}^t + \mathbb{E}V_{t+1}(q_t + m_t, \mathbf{S}^{t+1}(\mathbf{s}^t, 1)),$$

and the boundary condition is $V_{T+1}(q_{T+1}, \mathbf{s}^{T+1}) = -G(q_{T+1} - d)$. Here, $s_{[i]}^t$ denotes the ith largest element in $\mathbf{s}^t$, the expectation in V_t is with respect to $\mathbf{W}^t$ and $\mathbf{S}_B^{t+1}$, and the expectation in J_t is with respect to $\mathbf{S}_B^{t+1}$.

We define the optimal stopping rule as

$$a_t^*(q_t, \mathbf{s}^t) = \mathbf{1}\left(\max_{1\le m_t\le n_t} J_t(q_t, m_t, \mathbf{s}^t) \ge \mathbb{E}V_{t+1}(q_t, \mathbf{S}^{t+1}(\mathbf{s}^t, 0))\right),$$

where $\mathbf{1}(\cdot)$ is the indicator function, and the optimal number of offers to make as

$$m_t^*(q_t, \mathbf{s}^t) = \arg\max_{1\le m_t\le n_t} J_t(q_t, m_t, \mathbf{s}^t). \tag{2}$$

When there are multiple maximizers, $m_t^*(q_t, \mathbf{s}^t)$ is defined as the largest one. One can easily verify that a cutoff policy is optimal. In other words, in the same period, an applicant is hired only if all applicants with strictly higher scores are hired. Therefore, $s_{[m_t^*(q_t,\mathbf{s}^t)]}^t$ represents the optimal cutoff score. Let

$$M_t(q_t, \mathbf{s}^t) = \left\{i \in \{1, 2, \ldots, n_t\} : s_i^t \ge s_{[m_t^*(q_t,\mathbf{s}^t)]}^t\right\},$$

which represents the set of all applicants whose scores are greater than the cutoff score. However, $M_t(q_t, \mathbf{s}^t)$ is not necessarily the offer list because when multiple applicants have the same score as $s^t_{[m^*_t(q_t,\mathbf{s}^t)]}$, only some of them are on the offer list.

Before we investigate the optimal policy, we first establish the following properties of the value function, which are crucial for showing the optimal policy.

Lemma 1.

(i) $V_t(q_t, \mathbf{s}^t)$ *is convex increasing in* $\mathbf{s}^t$.

(ii) $\nabla_{s^t_j} V_t(q_t, \mathbf{s}^t) \leq 1$, $j = 1, 2, \ldots, n_t$.

(iii) $\nabla_{s^t_j} \mathbb{E} V_{t+1}(q_t, \mathbf{S}^{t+1}(\mathbf{s}^t, 0)) \leq 1 - p$, $j = 1, 2, \ldots, n_t$.

Lemma 1(i) provides the monotonicity and convexity of the value function. Lemma 1(ii) means that the marginal value of each score is always less than one. The interpretation for the third result is that when the recruiter waits, the marginal value of each score cannot exceed the probability of the applicant remaining in the system in the next period. As the dynamic program has a multiple-dimension state space, the optimal policy is complex. We characterize the optimal policy through answering the following questions. First, how does the optimal policy with respect to applicant j change if the score of applicant j increases? Second, how does the optimal policy with respect to applicant j change if the score of applicant i increases? Third, how can we extend the results in the previous two questions to any number of scores? Finally, what happens to the optimal policy if scores become more dispersed? We can form an overview of the optimal policy after these questions are answered.

4 Optimal Policy

We start by analyzing the problem of determining the optimal number of offers when the recruiter stops. Let $\mathbf{s} = (s_1, s_2, \ldots, s_n) \in \mathbb{R}^n_+$ and $s_{[i]}$ be the ith largest value in $\mathbf{s}$. Consider the following optimization problem:

$$f(\mathbf{s}) = \max_{1 \leq m \leq n} \left\{ \sum_{i=1}^{m} s_{[i]} + g(m) \right\}, \tag{3}$$

where $g(m)$ is any real-valued function of m. Determining the optimal number of offers when the recruiter stops in (1) is a special case of (3). Denote by $m^*(\mathbf{s})$ the largest maximizer in (3) and let $M(\mathbf{s}) = \left\{ i \in \{1, 2, \ldots, n\} : s_i \geq s_{[m^*(\mathbf{s})]} \right\}$. The following lemma provides useful properties of (3).

Lemma 2. *For any $j \in \{1, 2, \ldots, n\}$, the following statements about (3) hold.*

(i) The set $\left\{s_j \geq 0 : s_j \geq s_{[m^*(\mathbf{s})]}\right\}$ is nonempty and its minimum is attainable. Let $c_j(\mathbf{s}_{-j}) = \min\left\{s_j \geq 0 : s_j \geq s_{[m^*(\mathbf{s})]}\right\}$. There exist two constants $\underline{m}, \overline{m} \in \{1, 2, \ldots, n\}$ with $\underline{m} \leq \overline{m}$ such that $m^*(\mathbf{s}) = \underline{m}$ for all $s_j < c_j(\mathbf{s}_{-j})$ and $m^*(\mathbf{s}) = \overline{m}$ for all $s_j \geq c_j(\mathbf{s}_{-j})$. Moreover, $j \in M(\mathbf{s})$ if and only if $s_j \geq c_j(\mathbf{s}_{-j})$.

(ii) $f(\mathbf{s})$ as a function of s_j can be written as

$$f(\mathbf{s}) = (s_j - c_j(\mathbf{s}_{-j}))^+ + f(s_1, \ldots, s_{j-1}, 0, s_{j+1}, \ldots, s_n).$$

Lemma 2(i) introduces a value $c_j(\mathbf{s}_{-j})$, which is critical to our analysis throughout the paper. This value can be interpreted as the lowest score that an applicant must have to be on the offer list, given the scores of other applicants, if the recruiter decides to stop. The number of offers, $m^*(\mathbf{s})$, is a step function of s_j and it strictly increases only at $s_j = c_j(\mathbf{s}_{-j})$. Lemma 2($ii$) provides an explicit expression of $f(\mathbf{s})$ as a function of s_j. It can be seen that f is independent of s_j if s_j is below $c_j(\mathbf{s}_{-j})$ and is linearly increasing with slope 1 if s_j is above $c_j(\mathbf{s}_{-j})$. That is, s_j affects f if and only if applicant j is on the offer list.

Having established the properties of the maximal expected reward when the recruiter stops, we proceed to study the recruiter's optimal stopping policy. The following theorem summarizes the structure of the optimal stopping rule and the optimal number of offers to make with respect to each individual's score.

Theorem 1. *For $t = 1, 2, \ldots, T$, there exists a pair of thresholds $(L_j^t(q_t, \mathbf{s}_{-j}^t), U_j^t(q_t, \mathbf{s}_{-j}^t))$ with $-\infty \leq L_j^t(q_t, \mathbf{s}_{-j}^t) \leq U_j^t(q_t, \mathbf{s}_{-j}^t) \leq \infty$ for $j = 1, 2, \ldots, n_t$ such that the following holds:*

(i) The optimal stopping rule is

$$a_t^*(q_t, \mathbf{s}^t) = \begin{cases} 1 & \text{if } \ 0 \leq s_j^t \leq L_j^t(q_t, \mathbf{s}_{-j}^t), \\ 0 & \text{if } \ L_j^t(q_t, \mathbf{s}_{-j}^t) < s_j^t < U_j^t(q_t, \mathbf{s}_{-j}^t), \\ 1 & \text{if } \ s_j^t \geq U_j^t(q_t, \mathbf{s}_{-j}^t). \end{cases}$$

(ii) There exist two constants $\underline{m}_t, \overline{m}_t \in \{1, 2, \ldots, n_t\}$ with $\underline{m}_t \leq \overline{m}_t$ such that $m_t^*(q_t, \mathbf{s}^t) = \underline{m}_t$ for all $s_j^t \leq L_j^t(q_t, \mathbf{s}_{-j}^t)$ and $m_t^*(q_t, \mathbf{s}^t) = \overline{m}_t$ for all $s_j^t \geq U_j^t(q_t, \mathbf{s}_{-j}^t)$.

(iii) Applicant j is hired if and only if $s_j^t \geq U_j^t(q_t, \mathbf{s}_{-j}^t)$.

Theorem 1(i) shows that the optimal stopping rule is characterized by two threshold levels $L_j^t(q_t, \mathbf{s}_{-j}^t)$ and $U_j^t(q_t, \mathbf{s}_{-j}^t)$. That is, if applicant j's score is lower than $L_j^t(q_t, \mathbf{s}_{-j}^t)$, then the recruiter stops and makes no offers to j; if j's score is higher than $U_j^t(q_t, \mathbf{s}_{-j}^t)$, then the recruiter also stops but makes an offer to j; if the score is between $L_j^t(q_t, \mathbf{s}_{-j}^t)$ and $U_j^t(q_t, \mathbf{s}_{-j}^t)$, then the recruiter waits. Because Theorem 1(i) holds for any j, it implies that for any two applicants i and j, j's score falls within his or her waiting region $(L_j^t(q_t, \mathbf{s}_{-j}^t), U_j^t(q_t, \mathbf{s}_{-j}^t))$ if and only if i's score also falls within his or her waiting region $(L_i^t(q_t, \mathbf{s}_{-i}^t), U_i^t(q_t, \mathbf{s}_{-i}^t))$. This point will be clearer later in Theorem 2 where we discuss two applicants simultaneously.

It is expected that an applicant will be on the offer list if and only if their score is high enough. Somewhat surprising is the result that the recruiter may change from stopping to waiting as an applicant's score increases. For the recruiter, there is a trade-off between accepting qualified applicants already in the system (i.e., stopping) and waiting for potentially more qualified applicants to arrive at the risk of losing the qualified applicants already in the system. When an applicant has a very low score, the recruiter stops and makes an offer to other applicants with high scores. Otherwise, these applicants may no longer be available in the next period, and the recruiter will be left with applicants with very low scores. As the applicant's score increases, the risk of the recruiter being left with applicants with very low scores decreases. Therefore, the recruiter chooses to wait in the hope that more qualified applicants will arrive in the next period. Finally, if the applicant's score increases further, then an offer to the applicant is warranted, and the recruiter stops to make offers to the applicant as well as to other qualified applicants.

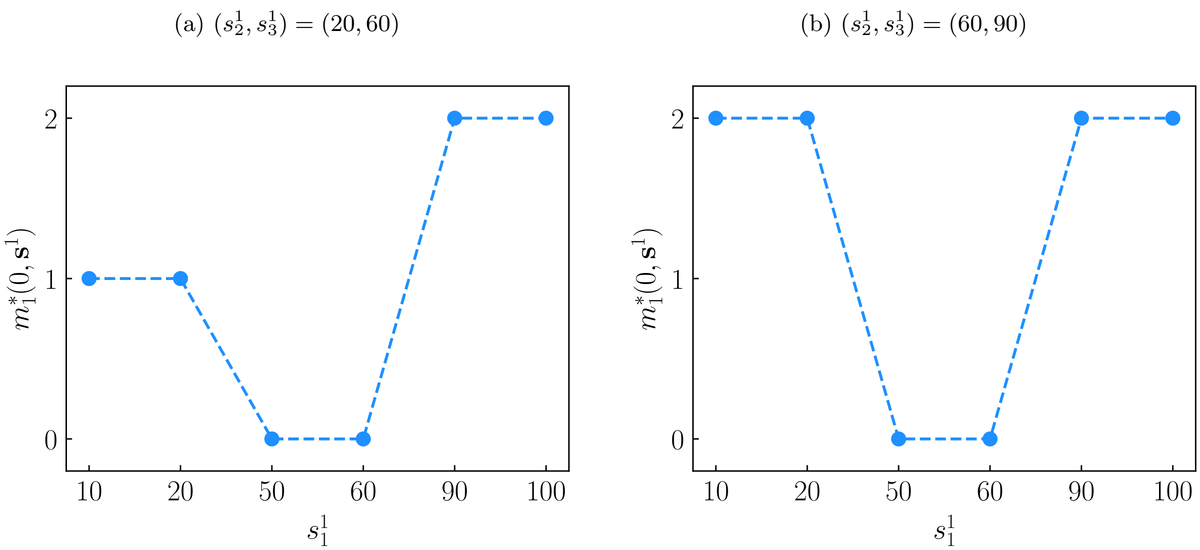
It would be useful to consider two special cases: $p = 0$ and $p = 1$. When $p = 0$, it is optimal for the recruiter to delay all decisions to the end. In this case, $L_j^t(q_t, \mathbf{s}_{-j}^t) = -\infty$ and $U_j^t(q_t, \mathbf{s}_{-j}^t) = \infty$. When $p = 1$, it is optimal for the recruiter to stop in every period. In this case, $L_j^t(q_t, \mathbf{s}_{-j}^t) = U_j^t(q_t, \mathbf{s}_{-j}^t)$. In other words, the transition from stopping to waiting when a score increases does not occur in these two special cases.

According to Theorem 1(ii), the optimal number of offers remains unchanged at $\underline{m}_t$ when s_j^t increases from zero to $L_j^t(q_t, \mathbf{s}_{-j}^t)$, and it remains unchanged at $\overline{m}_t$ when s_j^t increases from $U_j^t(q_t, \mathbf{s}_{-j}^t)$ to infinity. The value $\overline{m}_t$ can be the same as $\underline{m}_t$, in which case applicant j replaces another applicant on the offer list when s_j^t increases. It can also be strictly greater than $\underline{m}_t$. In Li and Yu (2021), where the recruiter must stop at a fixed sequence of time epochs, the impact of increasing a score on the total number of offers is bounded by one. This is not true in our setting. In some cases, the optimal number of offers can change by more than one. Li and Yu (2021) show that in their setting, there is a threshold policy to determine whether an applicant is on the offer list, and the threshold is independent of all scores in the same period if the changes in scores do not change the order. Similarly, Theorem 1(iii) shows that an applicant is on the offer list if and only if their score is higher than the threshold. The threshold $U_j^t(q_t, \mathbf{s}_{-j}^t)$, however, depends on the scores of all of the other applicants.

To better understand Theorem 1, let us consider the following example.

Example 1. Consider a horizon of three periods with exactly three applicants arriving in each period. Their scores follow a 6-point distribution: $\mathbb{P}(S_i^t = x_j) = P_j$, where x_j and P_j are the jth values in $(10, 20, 50, 60, 90, 100)$ and $(0.5, 0.2, 0.05, 0.05, 0.1, 0.1)$, respectively. Let $G(q_{T+1} - d) = u(q_{T+1} - d)^- + o(q_{T+1} - d)^+$ for $u = 10$ and some sufficiently large value o. In other words, overage (i.e., $q_{T+1} > d$) is not allowed. We consider two cases with the parameters in Figure 1. In the first case, the realized scores of applicants 2 and 3 in period 1 are $(20, 60)$, and in the second case, their scores are $(60, 90)$.

Figure 1: (Color online) Optimal Number of Offers for s_1^1



Notes. The parameters are $d = 3$ and $p = 0.1$.

Recall that the optimal number of offers to make in period t when the recruiter stops is defined as $m_t^*(q_t, \mathbf{s}^t)$ (see (2)). In Figure 1, we illustrate how $m_1^*(0, \mathbf{s}^1)$ changes with applicant 1's score, s_1^1. When the recruiter waits (i.e., $a_1^*(0, \mathbf{s}^1) = 0$), we let $m_1^*(0, \mathbf{s}^1) = 0$. The figures confirm the two-threshold stopping rule shown in Theorem 1(i). For example, in Figure 1(a), when $s_1^1 < 50$ or $s_1^1 > 60$, the recruiter stops; when $s_1^1 = 50$ or $s_1^1 = 60$, the recruiter waits. Furthermore, the figures show that in general, $m_1^*(0, \mathbf{s}^1)$ increases with s_1^1 when the recruiter stops and may strictly increase only when s_1^1 increases from below the lower threshold to above the higher threshold. In addition, applicant 1 is hired after his or her score surpasses 90 in both Figures 1(a) and 1(b). In summary, all of these observations are consistent with Theorem 1.

Interestingly, from Figure 1(b), we can see that when s_1^1 changes from 20 to 90, applicant 2 is disadvantaged, as he or she is replaced by applicant 1 on the offer list. However, when s_1^1 increases from 60 to 90, applicant 3 is better off, as the recruiter changes from waiting to stopping, and he or she is hired. This raises the question of what impact competitors becoming more qualified has on an applicant's likelihood of receiving an offer. Because when the score of an applicant changes, the optimal stopping rule is characterized by two thresholds, to answer the above question, we examine how the two thresholds depend on the score of another applicant, and the results will be shown in Theorem 2. We provide the following lemma, which generalizes Lemma 2 and is a building block for Theorem 2.

For ease of exposition, we first give some notations. As for Lemma 2(i), we define $c_i(\mathbf{s}_{-i}) = \min\{s_i \geq 0 : s_i \geq s_{[m^*(\mathbf{s})]}\}, i = 1, 2, \ldots, n$, where $m^*(\mathbf{s})$ is the largest maximizer in (3). For any $i, j \in \{1, 2, \ldots, n\}$ and $i > j$, let

$$\hat{\mathbf{s}} = (s_1, \ldots, s_{j-1}, 0, s_{j+1}, \ldots, s_{i-1}, s_i, s_{i+1}, \ldots, s_n),$$

$$\text{and} \quad \check{\mathbf{s}} = (s_1, \ldots, s_{j-1}, c_i(\hat{\mathbf{s}}_{-i}), s_{j+1}, \ldots, s_{i-1}, s_i, s_{i+1}, \ldots, s_n).$$

By definition, $c_i(\hat{\mathbf{s}}_{-i})$ is the lowest score that applicant i must have to be on the offer list, given other scores $\hat{\mathbf{s}}_{-i}$, if the recruiter decides to stop, and $c_i(\check{\mathbf{s}}_{-i})$ has a similar meaning. Both $c_i(\hat{\mathbf{s}}_{-i})$ and $c_i(\check{\mathbf{s}}_{-i})$ are independent of s_i and s_j. The two vectors $\hat{\mathbf{s}}$ and $\check{\mathbf{s}}$ are introduced for the purpose of defining the critical points $c_i(\hat{\mathbf{s}}_{-i})$ and $c_i(\check{\mathbf{s}}_{-i})$. They are constructed as follows. We start with $(s_1, \ldots, s_{j-1}, 0, s_{j+1}, \ldots, s_{i-1}, 0, s_{i+1}, \ldots, s_n)$, where both s_j and s_i are 0. We first increase s_j to $c_i(\hat{\mathbf{s}}_{-i})$, which is the lowest value required for j to be included on the offer list. We then increase s_i to the critical point $c_i(\check{\mathbf{s}}_{-i})$, which is the lowest value required for i to be included on the offer list.

Lemma 3. *For any $i, j \in \{1, 2, \ldots, n\}$ with $i > j$, the following statements about (3) hold.*

(i) $c_i(\hat{\mathbf{s}}_{-i}) \geq c_i(\check{\mathbf{s}}_{-i})$, and $f(\mathbf{s})$ as a function of s_j and s_i can be written as

$$f(\mathbf{s}) = \begin{cases} \left(s_j - \max\left\{\min\left\{\hat{b}, s_i\right\}, c_i(\hat{\mathbf{s}}_{-i})\right\}\right)^+ + f(\hat{\mathbf{s}}) & \text{if } c_i(\hat{\mathbf{s}}_{-i}) = c_i(\check{\mathbf{s}}_{-i}), \\ \left(s_j - \left(c_i(\hat{\mathbf{s}}_{-i}) - \min\left\{c_i(\hat{\mathbf{s}}_{-i}), s_i\right\} + \min\left\{c_i(\check{\mathbf{s}}_{-i}), s_i\right\}\right)\right)^+ + f(\hat{\mathbf{s}}) & \text{if } c_i(\hat{\mathbf{s}}_{-i}) > c_i(\check{\mathbf{s}}_{-i}), \end{cases}$$

where $\hat{b}$ is a constant with $\hat{b} \geq c_i(\hat{\mathbf{s}}_{-i}) = c_i(\check{\mathbf{s}}_{-i})$.

(ii) If $s_i \geq s_j$, then $f(\mathbf{s}+\delta\mathbf{e}_i) \geq f(\mathbf{s}+\delta\mathbf{e}_j)$ for any $\delta \geq 0$.

Lemma 2 shows that f as a function of s_j is piecewise linear with a breakpoint at $c_j(s_{-j})$. Lemma 3(i) further shows how the breakpoint depends on s_i and provides an explicit expression of $f(\mathbf{s})$ as a function of both s_j and s_i. This expression depends on whether $c_i(\hat{\mathbf{s}}_{-i}) = c_i(\check{\mathbf{s}}_{-i})$ or $c_i(\hat{\mathbf{s}}_{-i}) > c_i(\check{\mathbf{s}}_{-i})$. Lemma 3$(ii)$ shows that the marginal value of increasing a higher score is higher than that of increasing a lower score.

We are now ready to present our second main result about the optimal policy. The result shows how the two thresholds in Theorem 1 change when the score of another applicant changes. For any $i \in \{1, 2, \ldots, n_t\}$ and given $(q_t, \mathbf{s}^t_{-i})$, we define $c^t_i(q_t, \mathbf{s}^t_{-i}) = \min\left\{s^t_i \geq 0 : s^t_i \geq s^t_{[m^*_t(q_t,\mathbf{s}^t)]}\right\}$. We define $(\hat{\mathbf{s}}^t, \check{\mathbf{s}}^t, \hat{b}_t)$ similarly as we do $(\hat{\mathbf{s}}, \check{\mathbf{s}}, \hat{b})$ in Lemma 3, respectively.

Theorem 2. *For $t = 1, 2, \ldots, T$ and $i, j \in \{1, 2, \ldots, n_t\}$ that satisfies $i > j$, let $\beta = \hat{b}_t$ if $c^t_i(q_t, \hat{\mathbf{s}}^t_{-i}) = c^t_i(q_t, \check{\mathbf{s}}^t_{-i})$, and $\beta = c^t_i(q_t, \check{\mathbf{s}}^t_{-i})$ if $c^t_i(q_t, \hat{\mathbf{s}}^t_{-i}) > c^t_i(q_t, \check{\mathbf{s}}^t_{-i})$. Then, $\lim_{s^t_i \to \infty} L^t_j(q_t, \mathbf{s}^t_{-j}) = \lim_{s^t_i \to \infty} U^t_j(q_t, \mathbf{s}^t_{-j}) = \beta$, and the following is true.*

(i) $L^t_j(q_t, \mathbf{s}^t_{-j})$ is continuously decreasing for $s^t_i \leq c^t_i(q_t, \hat{\mathbf{s}}^t_{-i})$ and continuously increasing for $s^t_i \geq c^t_i(q_t, \hat{\mathbf{s}}^t_{-i})$.

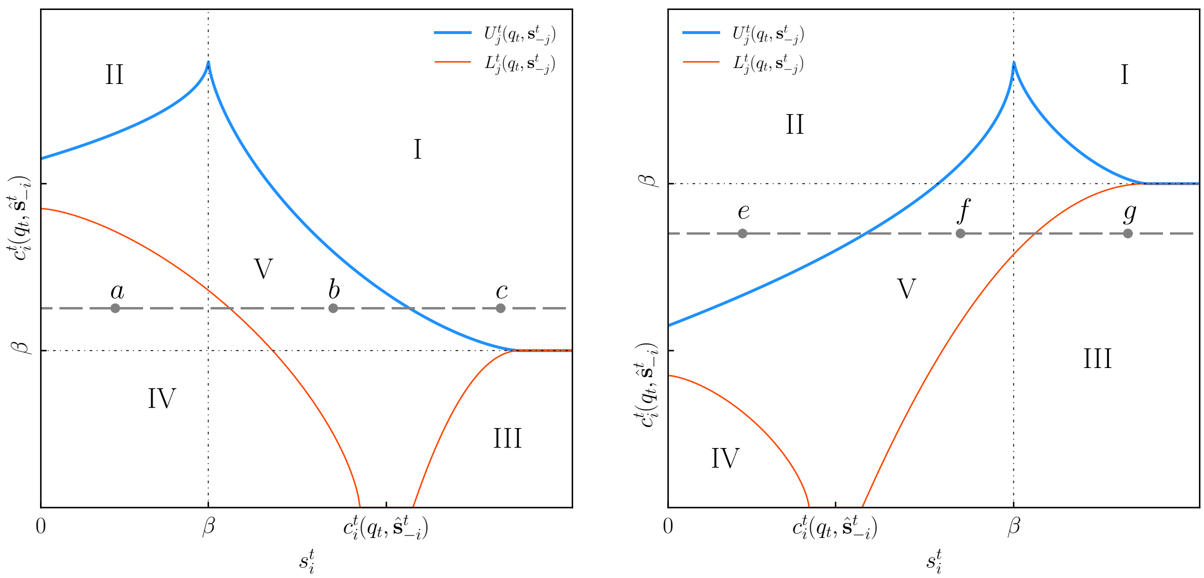
(ii) $U^t_j(q_t, \mathbf{s}^t_{-j})$ is continuously increasing for $s^t_i \leq \beta$ and continuously decreasing for $s^t_i \geq \beta$.

The results presented in Theorem 2 are visualized in Figure 2. Figure 2(a) refers to the case in which $c^t_i(q_t, \hat{\mathbf{s}}^t_{-i}) > c^t_i(q_t, \check{\mathbf{s}}^t_{-i})$, and Figure 2(b) refers to the case in which $c^t_i(q_t, \hat{\mathbf{s}}^t_{-i}) = c^t_i(q_t, \check{\mathbf{s}}^t_{-i})$. In each figure, the upper dark curve represents the upper threshold $U^t_j(q_t, \mathbf{s}^t_{-j})$ for applicant j as a function of i's score s^t_i, and the lower light curve denotes the lower threshold $L^t_j(q_t, \mathbf{s}^t_{-j})$ for j as a function of s^t_i. Both are symmetric around the 45-degree line, because interchanging the ith and jth elements of the score vector $\mathbf{s}^t$ does not alter the analysis. Thus, if we treat the y-axis (representing j's score s^t_j) as the x-axis, we obtain the upper and lower curves for i, which behave identically to those of j. According to Theorem 1(i), there are five scenarios for each of the two cases described above. For example, in Figure 2(a), (1) if the score pair (s^t_i, s^t_j) falls into region I, both applicants j and i are hired; (2) if (s^t_i, s^t_j) is in region II, applicant j is hired but i is not; (3) if (s^t_i, s^t_j) is in region III, applicant i is hired but j is not; (4) if (s^t_i, s^t_j) is in region IV, the recruiter chooses to stop but neither applicant j nor i is hired; and (5) if (s^t_i, s^t_j) is in region V, the recruiter chooses to wait. Figure 2(b) can be similarly explained.

Theorem 2 offers additional insights that are not in Theorem 1. First, Figure 2 illustrates that an applicant is hired if and only if his or her score is sufficiently high to exceed the upper

Figure 2: (Color online) Optimal Policy for Hiring Applicants j and i

(a) If $c_i^t(q_t, \hat{\mathbf{s}}_{-i}^t) > c_i^t(q_t, \check{\mathbf{s}}_{-i}^t)$ (b) If $c_i^t(q_t, \hat{\mathbf{s}}_{-i}^t) = c_i^t(q_t, \check{\mathbf{s}}_{-i}^t)$

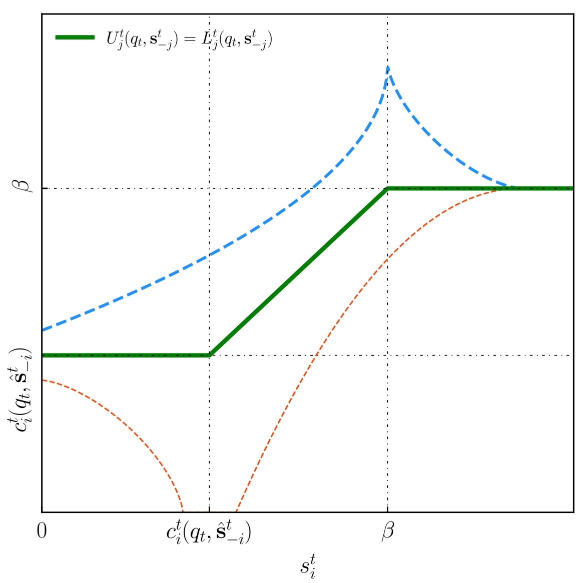


dark curve, which is consistent with Theorem 1. Moreover, the upper threshold is quasi-concave in the competitor's score. Second, for any given s_j^t, there exist two thresholds such that applicant j is hired if and only if s_i^t is either greater than the larger threshold or smaller than the smaller threshold. Third, from region V, if one applicant's score, say j's, falls within their waiting region $(L_j^t(q_t, \mathbf{s}_{-j}^t), U_j^t(q_t, \mathbf{s}_{-j}^t))$, all other applicants' scores must also fall within their respective waiting regions. Fourth, as demonstrated in Theorem 1(ii), when the score pair (s_i^t, s_j^t) changes within each region in Figure 2, the optimal number of offers remains unchanged. Among regions $\{\text{I}, \text{II}, \text{III}, \text{IV}\}$, the optimal number of offers is highest in I and lowest in IV. The optimal numbers of offers in II and III are the same, and are in the middle.

Figure 2 also presents six cases in which applicant i's score increases while j's score is kept unchanged. In Figure 2(a), (a1) when the scores change from a to b, the recruiter will change from stopping to waiting; (a2) when (s_i^t, s_j^t) moves from point b to c, the recruiter will change from waiting to stopping and hire both applicants j and i; and (a3) when (s_i^t, s_j^t) moves from point a to c, applicant j, who was initially not hired, will be hired. In Figure 2(b), (b1) when (s_i^t, s_j^t) moves from point e to f, the recruiter, who initially stopped with applicant j on the offer list, will now wait, and hence, no one is hired; (b2) when (s_i^t, s_j^t) moves from point f to g, the recruiter will stop and hire applicant i but not j; and (b3) when (s_i^t, s_j^t) moves from point e to g, applicant j, who was initially on the offer list, will be replaced by i.

These cases together reveal several interesting findings. Does an applicant's likelihood of being hired increase or decrease when another applicant's score increases? Cases (b1) ((s_i^t, s_j^t) moves from e to f) and (b3) (e to g) illustrate that applicant j is indeed disadvantaged by an increase in applicant i's score. This is expected because they are competing for a limited number of jobs. However, from case (a2) (b to c) and (a3) (a to c), we also find that an applicant may actually benefit from an increase in another applicant's score. Such a "free riding" phenomenon is somewhat counter-intuitive. In case (a2), recruiter switches from waiting to stopping and hiring both. Once the recruiter decides to stop in the current period, they may need to wait longer in the following periods for the market to become sufficiently thick again. As a result, the recruiter is willing to lower their standards to a level that applicant j can meet. In case (a3), recruiter switches from stopping and hiring neither of them to stopping and hiring both. The recruiter needs to consider both hiring qualified applicants and meeting the target at the same time. Hiring both of them, as opposed to hiring only one of them, moves the total number of hired applicants closer to the target. As a result, the recruiter can afford to be more patient in waiting for qualified applicants in subsequent periods.

Figure 3: (Color online) Optimal Policy for Hiring Applicants j and i When $p = 1$



Notes. When $p = 1$, $c_i^t(q_t, \hat{\mathbf{s}}_{-i}^t) = c_i^t(q_t, \check{\mathbf{s}}_{-i}^t)$ always holds.

To see why the potential waiting is the real driver behind the free riding phenomenon, let us look at the special case when $p = 1$. In this case, as applicants stay in the systems for only one period, it is optimal to stop every period. It is easy to show that $c_i^t(q_t, \hat{\mathbf{s}}_{-i}^t) = c_i^t(q_t, \check{\mathbf{s}}_{-i}^t)$. Figure 3 illustrates how the optimal policy for applicant j changes with i's score. This figure is a special case of Figure 2(b). In the figure, $L_j^t(q_t, \mathbf{s}_{-j}^t)$ and $U_j^t(q_t, \mathbf{s}_{-j}^t)$ coincide along the solid curve, causing the waiting region to shrink to the empty set. Therefore, applicant j is hired if s_j^t is above the solid curve, and rejected otherwise. Additionally, the curve increases with s_i^t, suggesting that the two applicants are competing and that applicant j cannot benefit from a more qualified applicant i. In other words, when $p = 1$, the free riding phenomenon does not occur.

Theorems 1 and 2 can be generalized to any $n \in \{1, 2, \ldots, n_t\}$ scores. Although not as intuitive as Theorems 1 and 2, the following theorem gives a "global view" of the optimal policy in general. Let δ_i^t represent whether applicant $i \in \{1, 2, \ldots, n\}$ is hired. It is equal to one if i is hired and zero otherwise.

Theorem 3. *There exists a unique collection of sets* $\{\mathcal{P}_{\mathcal{I}}^t\}_{\mathcal{I} \subset \{1,2,\ldots,n\}}$ *in* $\mathbb{R}_+^n$ *such that it is pairwise disjoint and each* $\mathcal{P}_{\mathcal{I}}^t$ *is connected. Let* $\mathcal{C}^t = \mathbb{R}_+^n \setminus \bigcup_{\mathcal{I} \subset \{1,2,\ldots,n\}} \mathcal{P}_{\mathcal{I}}^t$. *Any ray* $\left\{(s_i^t)_{i \in \{1,2,\ldots,n\}} : s_j^t \geq 0\right\}$ *in* $\mathbb{R}_+^n$ *can sequentially intersect at most three sets* $\mathcal{P}_{\mathcal{I}}^t$, $\mathcal{C}^t$ *and* $\mathcal{P}_{\mathcal{J}}^t$. *Furthermore, the following statements hold.*

(i) If $(s_i^t)_{i \in \{1,2,\ldots,n\}} \in \mathcal{C}^t$, *then* $a_t^*(q_t, \mathbf{s}^t) = 0$.

(ii) If $(s_i^t)_{i \in \{1,2,\ldots,n\}} \in \mathcal{P}_{\mathcal{I}}^t$, *then* $\delta_i^t = 1$ *for all* $i \in \mathcal{I}$ *and* $\delta_i^t = 0$ *for all* $i \in \{1, 2, \ldots, n\} \setminus \mathcal{I}$. *In addition,* δ_i^t *is constant for any* $i \in \{n+1, n+2, \ldots, n_t\}$ *on* $\mathcal{P}_{\mathcal{I}}^t$.

(iii) There exists an increasing function $\mathcal{M}_t : \{0, 1, \ldots, n\} \to \{0, 1, \ldots, n_t\}$ *such that* $m_t^*(q_t, \mathbf{s}^t) = \mathcal{M}_t(m)$ *for any* $(s_i^t)_{i \in \{1,2,\ldots,n\}} \in \mathcal{P}_{\mathcal{I}}^t$ *with* $|\mathcal{I}| = m$.

Theorem 3 demonstrates that the score space $\mathbb{R}_+^n$ can be partitioned into at most $2^n + 1$ nonempty sets. In particular, the recruiter waits when the n scores fall within region $\mathcal{C}^t$; otherwise, the recruiter stops. The waiting region $\mathcal{C}^t$ is positioned "centrally", in the sense that it lies between two stopping regions along any axis in the score space. Within each stopping region $\mathcal{P}_{\mathcal{I}}^t$, for applicants in $\{1, 2, \ldots, n\}$, only those in $\mathcal{I}$ can be hired; the hiring status of applicants not in $\{1, 2, \ldots, n\}$ remains unchanged. Moreover, due to the permutation invariant of the optimal policies, there are $\binom{n}{|\mathcal{I}|}$ regions corresponding to exactly $|\mathcal{I}|$ applicants in $\{1, 2, \ldots, n\}$ to be hired. The optimal number of total offers to make is the same across all these $\binom{n}{|\mathcal{I}|}$ regions and increases with $|\mathcal{I}|$. Therefore, when $n = n_t$, the optimal number of offers to make and the specific applicants to whom offers are made can be fully characterized. As illustrated in Theorems 1 and 2, when $n = 1$, there are

$2^1 + 1 = 3$ regions: $\mathcal{P}^t_\emptyset$ represents the lower region, $\mathcal{P}^t_{\{1\}}$ corresponds to the upper region, and $\mathcal{C}^t$ denotes the middle waiting region. For $n = 2$, there are $2^2 + 1 = 5$ regions. The meaning of these regions is also consistent with Theorem 3, as shown in Figure 2.

5 Impact of Score Dispersion

What happens to the optimal policy or the total expected reward of the recruiter if the current scores are more dispersed? The following theorem examines how the optimal policy and the recruiter's reward are affected if two applicants' scores in the current period get closer to each other while the mean is kept unchanged.

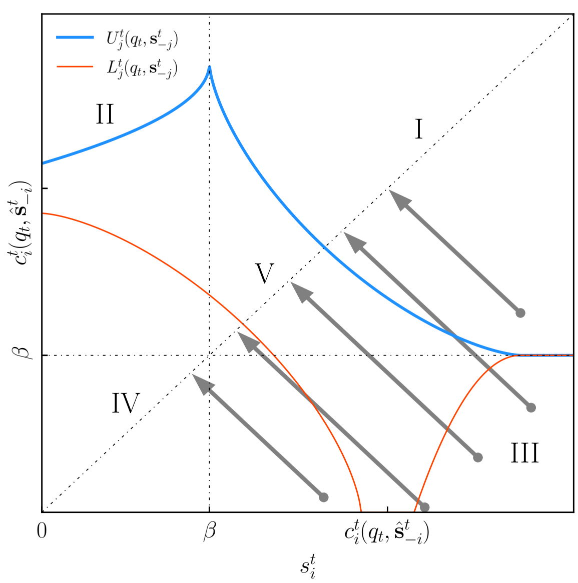
Theorem 4. *For any $(q_t, \mathbf{s}^t)$ with $s^t_i > s^t_j$ and $\delta > 0$ such that $s^t_i - \delta \geq s^t_j + \delta$, we have the following:*

(i) $V_t(q_t, \mathbf{s}^t) \geq V_t(q_t, \mathbf{s}^t + \delta \mathbf{e}_j - \delta \mathbf{e}_i)$.

(ii) If $i, j \in M_t(q_t, \mathbf{s}^t)$, then $i, j \in M_t(q_t, \mathbf{s}^t + \delta \mathbf{e}_j - \delta \mathbf{e}_i)$ and $V_t(q_t, \mathbf{s}^t) = V_t(q_t, \mathbf{s}^t + \delta \mathbf{e}_j - \delta \mathbf{e}_i)$.

(iii) If $a^*_t(q_t, \mathbf{s}^t) = 1$ and $i, j \notin M_t(q_t, \mathbf{s}^t)$, then $a^*_t(q_t, \mathbf{s}^t + \delta \mathbf{e}_j - \delta \mathbf{e}_i) = 1$, $i, j \notin M_t(q_t, \mathbf{s}^t + \delta \mathbf{e}_j - \delta \mathbf{e}_i)$, and $V_t(q_t, \mathbf{s}^t) = V_t(q_t, \mathbf{s}^t + \delta \mathbf{e}_j - \delta \mathbf{e}_i)$.

Figure 4: (Color online) Optimal Policy for Hiring Applicants j and i When s^t_i and s^t_j Are Closer



Notes. We use Figure 2(a) as an example to illustrate.

According to Theorem 4(i), in general, the recruiter benefits from a more diverse pool of applicants. This is expected because they only hire the top applicants, and the more diverse the scores are, the more qualified the top applicants are. Theorem 4(ii) and (iii) show that if two applicants are on the offer list or if the recruiter stops but neither applicant is on the offer list, then bringing their scores closer will not affect the recruiter's optimal policy. In this case, the recruiter's payoff also remains the same because a closer score pair does not alter the total score on the offer list. However, in other situations, the recruiter may be worse off, and whether the two applicants are hired depends on the context. Specifically, as illustrated in Figure 4, if applicant i is hired but j is not (region III), bringing their scores closer can result in four outcomes: (1) i is still on the offer list and j is not; (2) the recruiter waits; (3) the recruiter hires both i and j; or (4) the recruiter stops, but neither i nor j is hired. Similarly, if the recruiter decides to wait (region V), a closer score pair can lead to three scenarios: (1) the recruiter continues to wait; (2) the recruiter hires both i and j; or (3) the recruiter stops, but neither i nor j is hired.

The recruiter also benefits from a pool of applicants in the future with more variable scores or stochastically larger scores. To measure score variability, we consider the *convex order* from the theory of stochastic comparisons, which is commonly used in the operations literature in studies of the impact of variability (e.g., Lu et al. 2003, Levi et al. 2024). If $\tilde{\mathbf{S}}^t$ is larger than $\mathbf{S}^t$ in convex order, meaning that $\mathbb{E}\phi(\tilde{\mathbf{S}}^t) \geq \mathbb{E}\phi(\mathbf{S}^t)$ for any n_t-dimensional convex function ϕ, then the total expected reward of the recruiter is higher under the distribution of $\tilde{\mathbf{S}}^t$. Roughly speaking, $\tilde{\mathbf{S}}^t$ is more likely than $\mathbf{S}^t$ to take on "extreme" values over the support.

6 Numerical Studies

In this section, we conduct numerical studies to complement our theoretical analysis and quantify the benefit of endogenously determining the optimal times to stop and make offers. We define the value of waiting as the difference in the recruiter's total expected reward in our model and the expected reward when the recruiter must stop in every period. We first test the exact value of waiting under some simple parametric settings (including short horizons, known numbers of applicants, and discrete and finite score distributions). For more general and realistic scenarios (longer horizons, continuous score distributions, and random arrivals), exact evaluation is impossible. Therefore, we propose an easily implementable heuristic based on our theoretical findings, and we utilize it to estimate the value of waiting. For all of the numerical studies, we set the penalty cost to $G(x) = ux^{-} + ox^{+}$.

6.1 Exact Evaluation

In this subsection, we measure the value of waiting. The model in which the recruiter must stop in every period is studied by Li and Yu (2021). The dynamic programming formulation for that model is as follows:

$$V'_t(q_t, \mathbf{s}^t) = \max_{0 \le m_t \le n_t} \left\{ \sum_{i=1}^{m_t} s^t_{[i]} + \mathbb{E}V'_{t+1}(q_t + m_t, \mathbf{S}^{t+1}(\mathbf{s}^t, 1)) \right\},$$

where the boundary condition is $V'_{T+1}(q_{T+1}) = -G(q_{T+1} - d)$.

Table 1: Value of Waiting

	$d=1$				$d=3$				$d=5$			
p	0.01	0.10	0.30	0.60	0.01	0.10	0.30	0.60	0.01	0.10	0.30	0.60
$\mathbf{P} = (0.80, 0.15, 0.05)$												
$\mathbf{X} = (1, 20, 100)$	7.89	7.21	5.32	2.46	4.79	2.83	1.70	0.85	0.60	0.35	0.22	0.11
$\mathbf{X} = (1, 50, 100)$	10.78	9.29	5.26	1.51	2.99	1.60	0.64	0.18	0.30	0.14	0.03	0.01
$\mathbf{X} = (1, 80, 100)$	5.95	4.24	1.78	0.54	0.82	0.23	0.03	0.00	0.07	0.01	0.00	0.00
$\mathbf{X} = (20, 50, 80)$	7.29	6.27	3.54	0.99	1.81	0.96	0.38	0.10	0.16	0.07	0.02	0.00
$\mathbf{P} = (0.70, 0.20, 0.10)$												
$\mathbf{X} = (1, 20, 100)$	2.13	2.02	1.61	0.81	4.90	2.87	1.67	0.67	1.51	0.76	0.42	0.14
$\mathbf{X} = (1, 50, 100)$	4.69	4.41	3.40	1.44	6.02	3.39	1.90	0.87	1.41	0.79	0.42	0.18
$\mathbf{X} = (1, 80, 100)$	3.91	3.49	2.24	0.66	2.64	1.18	0.50	0.14	0.48	0.19	0.05	0.01
$\mathbf{X} = (20, 50, 80)$	3.41	3.20	2.46	1.02	4.04	2.27	1.26	0.56	0.88	0.49	0.26	0.11
$\mathbf{P} = (1/3, 1/3, 1/3)$												
$\mathbf{X} = (1, 20, 100)$	0.01	0.01	0.01	0.00	0.12	0.10	0.09	0.05	0.38	0.27	0.21	0.10
$\mathbf{X} = (1, 50, 100)$	0.01	0.01	0.01	0.01	0.34	0.28	0.21	0.12	0.99	0.72	0.40	0.19
$\mathbf{X} = (1, 80, 100)$	0.02	0.02	0.02	0.01	0.32	0.25	0.17	0.07	0.69	0.47	0.25	0.08
$\mathbf{X} = (20, 50, 80)$	0.01	0.01	0.01	0.01	0.26	0.22	0.16	0.09	0.74	0.54	0.30	0.14

Notes. % VoWs.

To evaluate the value of waiting, we try various levels of deterministic arrivals, point-type score distributions, departure probabilities, and hiring targets. Different combinations yield similar patterns in the numerical studies, and we therefore only report the results with the parameters in Table 1. Specifically, we consider a horizon of five periods with exactly three applicants arriving in each period, and their scores follow a 3-point distribution: $\mathbb{P}(S^t_i = x_j) = P_j$, where x_j and

P_j are the jth values in $\mathbf{X} = (x_1, x_2, x_3)$ and $\mathbf{P} = (P_1, P_2, P_3)$, respectively. The values taken by the system parameters are $u = 10$, o is sufficiently large such that overage is not allowed, $d \in \{1, 3, 5\}$, $p \in \{0.01, 0.1, 0.3, 0.6\}$, $\mathbf{X} \in \{(1, 20, 100), (1, 50, 100), (1, 80, 100), (20, 50, 80)\}$, and $\mathbf{P} \in \{(0.8, 0.15, 0.05), (0.7, 0.2, 0.1), (1/3, 1/3, 1/3)\}$.

We measure the value of waiting by the following expression:

$$\text{VoW} = \frac{\mathbb{E}V_1(0, \mathbf{S}^1) - \mathbb{E}V_1'(0, \mathbf{S}^1)}{\mathbb{E}V_1'(0, \mathbf{S}^1)} \times 100\%,$$

where VoW is the *value of waiting*. As we observe from Table 1, the VoW can be as high as 10.78%. Waiting generates considerable benefits when highly qualified applicants arrive less frequently than less qualified applicants (e.g., $\mathbf{P} = (0.8, 0.15, 0.05)$ or $(0.7, 0.2, 0.1)$), the probability that applicants depart is low (e.g., $p = 0.01, 0.1$, and 0.3), and the hiring target is low (e.g., $d = 1$ or 3). Under these conditions, the recruiter can effectively exercise the options of waiting.

6.2 Approximation

So far, we have accurately tested the value of waiting under simple parametric settings. Evaluating the value of waiting in more realistic settings, however, can be computationally challenging because of the large dimensions of the score space. For example, to compute the total reward for a simple case with a k-point score distribution and exactly n arrivals in each period, the dimension of the state space can be as large as k^{nT}, which is only feasible for small (k, n, T). In this subsection, we focus on approximations. To this end, we first propose a threshold-based heuristic that is easy to implement, is easy to compute, and performs better than some known heuristics in the literature.

6.2.1 Threshold-based Heuristic

Our results and analysis in Section 4 show that the optimal stopping rule is characterized by two thresholds. A key implication of the stopping rule is that when there are enough applicants whose scores are intermediate, the risk of waiting is small because the probability of being left with only applicants with very low scores is small. Therefore, the recruiter waits and hopes that more qualified applicants will arrive in the next period. Our proposed threshold-based heuristic is directly inspired by this idea.

Consider a system state $(q_t, \mathbf{s}^t)$ in period t. Let u_t be the solution of

$$\mathbb{E}\left[\sum_{i=1}^{N_t^B} \mathbf{1}(S_{B,i}^t \geq u_t)\right] = \frac{(d - q_t)^+}{T - t + 1}, \tag{4}$$

where N_t^B is the size of $\mathbf{S}_B^t$, which was defined in Section 3, and $S_{B,i}^t$ is the ith element in $\mathbf{S}_B^t$. Denote $\lambda_t = \mathbb{E}N_t^B$ (we assume that it is finite). We set $u_t = 0$ if $\lambda_t < (d - q_t)^+/(T - t + 1)$. The right side of Equation (4) is the average number of applicants that need to be hired in each remaining period, and the left side is the expected number of applicants whose scores are higher than u_t. Therefore, the solution u_t of Equation (4) can be interpreted as the minimum requirement for applicants to be on the offer list or as the threshold for the recruiter to identify high scores.

Next, let l_t be the solution of

$$\mathbb{E}\left[\sum_{i=1}^{N_t^B} \mathbf{1}(l_t < S_{B,i}^t < u_t)\right] = \frac{(d - q_t)^+}{T - t + 1},$$

where we set $l_t = 0$ if $\lambda_t < 2(d - q_t)^+/(T - t + 1)$. The left side of the above equation is the expected number of applicants whose scores are within (l_t, u_t). We consider scores in (l_t, u_t) as intermediate scores.

Let $K_t = (d - q_t)^+/(T - t + 1)$. By some simple algebra, we have that u_t and l_t are the $(1 - K_t/\lambda_t)^+$th and $(1 - 2K_t/\lambda_t)^+$th quantiles of the score distribution function F, respectively:

$$u_t = \inf\left\{s : F(s) \geq \left(1 - \frac{K_t}{\lambda_t}\right)^+\right\} \quad \text{and} \quad l_t = \inf\left\{s : F(s) \geq \left(1 - \frac{2K_t}{\lambda_t}\right)^+\right\}.$$

The threshold-based heuristic that we propose works as follows:

Step 1. Count the number of applicants whose scores are higher than u_t, i.e., $n_t^u = \sum_{i=1}^{|\mathbf{s}^t|} \mathbf{1}(s_i^t \geq u_t)$. If $n_t^u \leq K_t$, move to Step 2. If $n_t^u > K_t$, accept applicants whose scores are higher than u_t in descending order until d is met. After d is met, accept the remaining applicants if and only if their scores are higher than the marginal overage cost. Then, move to Step 3.

Step 2. Count the number of applicants whose scores are within (l_t, u_t), i.e., $n_t^l = \sum_{i=1}^{|\mathbf{s}^t|} \mathbf{1}(l_t < s_i^t < u_t)$. If either $n_t^l > K_t/(1-p)^2$ or $n_t^u = 0$ and d is not met, wait; otherwise, accept applicants whose scores are higher than u_t in descending order until d is met. After d is met, accept the remaining applicants if and only if their scores are higher than the marginal overage cost. Then, move to Step 3.

Step 3. Repeat the above two steps until period T. At period T, accept applicants greedily in descending order until d is met. Then, accept the remaining applicants if and only if their scores are higher than the marginal overage cost.

The thresholds l_t and u_t and K_t are all independent of the current scores. This greatly reduces the computational burden. In addition, the proposed heuristic uses the current score information to determine whether to stop or to wait based on (l_t, u_t, K_t). If there are enough applicants with high scores (i.e., $n_t^u > K_t$), the recruiter hires all of them as long as the target is not met. If the number of applicants with high scores is insufficient (i.e., $n_t^u \leq K_t$) but the number of applicants with intermediate scores is adequate (i.e., $n_t^l > K_t/(1-p)^2$), the recruiter waits. If the number of applicants with intermediate scores is also insufficient (i.e., $n_t^l \leq K_t/(1-p)^2$), the recruiter hires all applicants with high scores until the target is met as long as $n_t^u > 0$. Here we add a term $(1-p)^2$ for the waiting decision to take departures into account. Therefore, the recruiter will be less likely to wait if the departure probability is higher. Finally, because u_t decreases with K_t, the average number of vacancies to be filled per period, when K_t is larger, the recruiter lowers the standard (i.e., smaller u_t).

6.2.2 Approximated Value of Waiting

In this subsection, we examine the value of waiting using the threshold-based heuristic that we proposed earlier in more general parametric settings. For the model without waiting, we similarly introduce a simple single-threshold heuristic to compute the reward.

Table 2: Approximated Value of Waiting

	$T=5$								$T=8$							
	$p=0.1$				$p=0.3$				$p=0.1$				$p=0.3$			
λ	2	4	6	8	2	4	6	8	2	4	6	8	2	4	6	8
$\sigma=30$																
$d=2$	7.03	2.39	1.16	0.88	6.36	1.91	1.08	0.63	4.62	1.42	0.89	0.58	4.51	1.34	0.95	0.62
$d=4$	6.52	1.90	0.64	0.36	4.82	1.74	0.63	0.37	4.72	1.47	0.82	0.40	3.55	1.03	0.60	0.42
$d=6$	4.70	2.29	0.57	0.10	3.13	1.65	0.66	0.17	3.86	1.47	0.47	0.25	3.02	1.18	0.46	0.19
$d=8$	2.70	2.18	0.61	0.13	2.07	1.60	0.71	0.16	3.68	1.41	0.44	0.04	2.52	1.10	0.24	0.10
$d=10$	1.19	2.18	0.91	0.09	0.92	1.17	0.58	0.14	3.29	1.46	0.34	0.05	2.10	1.00	0.32	0.10
$\sigma=50$																
$d=2$	6.87	2.58	1.88	1.57	6.11	2.29	1.67	1.33	4.31	2.15	1.91	2.00	3.78	1.92	1.48	1.49
$d=4$	4.72	1.97	0.91	0.72	3.87	1.70	0.84	0.57	3.96	1.63	1.06	1.04	3.46	1.51	0.86	0.80
$d=6$	3.32	1.57	0.74	0.38	2.51	1.58	0.70	0.28	3.34	1.24	0.67	0.52	2.61	1.22	0.62	0.39
$d=8$	1.71	1.61	0.55	0.10	1.42	1.29	0.53	0.18	2.52	1.26	0.56	0.17	2.12	1.01	0.37	0.24
$d=10$	0.79	1.33	0.39	0.15	0.65	0.95	0.38	0.15	1.88	1.08	0.46	0.19	1.71	0.78	0.26	0.08

Notes. The results are based on 5,000 random samples. % $\widetilde{\text{VoW}}$s.

The heuristic for the model without waiting is a single-threshold policy governed by u_t that solves Equation (4). In each period, the recruiter only hires applicants whose scores are higher than u_t in descending order. After the target d is met, the recruiter accepts applicants if and only if their scores are higher than the marginal overage cost.

We assume that the number of applicants arriving in each period follows a Poisson distribution with rate λ and that their scores follow a normal distribution with mean μ and standard deviation σ. The system parameters are set to the following values: $u = 100$, $o = 180$, $\mu = 100$, $\sigma \in \{30, 50\}$, $T \in \{5, 8\}$, $p \in \{0.1, 0.3\}$, $\lambda \in \{2, 4, 6, 8\}$, and $d \in \{2, 4, 6, 8, 10\}$. We generate 5,000 random samples and compute the average rewards for the two heuristics. In total, 160 instances are reported.

We measure the approximated value of waiting by the following expression:

$$\widetilde{\mathrm{VoW}} = \frac{\mathbb{E}V_1^{\mathrm{TB}}(0, \mathbf{S}^1) - \mathbb{E}V_1^{\mathrm{ND}}(0, \mathbf{S}^1)}{\mathbb{E}V_1^{\mathrm{ND}}(0, \mathbf{S}^1)} \times 100\%,$$

where $\widetilde{\mathrm{VoW}}$ is the *approximated value of waiting* and V_1^{TB} and V_1^{ND} represent the maximal total rewards under the threshold-based heuristic for our model and the single-threshold heuristic for the model without waiting, respectively.

Table 3: Average Number of Periods Waited

	$T = 5$		$T = 8$	
p	0.1	0.3	0.1	0.3
$d = 2$	1.97	2.01	3.63	3.75
$d = 4$	1.75	1.68	3.51	3.55
$d = 6$	1.52	1.31	3.17	3.12
$d = 8$	1.27	0.98	2.98	2.71
$d = 10$	1.04	0.73	2.65	2.29

Table 2 presents the approximated value of waiting $\widetilde{\mathrm{VoW}}$ for the 160 instances. Overall, the $\widetilde{\mathrm{VoW}}$ is higher if the target is lower and the departure probability is lower, which is consistent with the results in Section 6.1. In addition, the $\widetilde{\mathrm{VoW}}$ is higher when the arrival rate is lower because when the arrival rate is low, it is harder to hire qualified applicants as soon as they arrive in a period and the need for waiting is greater. In Table 3, we report the average number of periods that the recruiter chooses to wait over the season (among the 5,000 randomly generated samples and the eight groups of (λ, σ) under each combination of (T, p, d)). The recruiter chooses to wait more if the recruitment season is longer, the hiring target is lower, and the departure probability is lower. Essentially, under these conditions, the recruiter has less urgency to extend offers.

Finally, we compare the performance of our threshold-based heuristic with other heuristics (e.g., those suggested in Du et al. (2024)), and we find that our heuristic not only is easier to compute but also performs better in general.

7 Waiting Lists

In our previous analysis, we made the assumption that once the recruiter stops, applicants who are not accepted depart immediately and cannot be recalled later. In this subsection, we consider an extension where applicants who are not accepted will not depart immediately. Instead, they are put on a waiting list and can be considered at a later time.

Being placed on a waiting list changes an applicant's trade-off between waiting and pursuing other options. We assume that applicants stay on waiting lists for at most k rounds, where the periods between two adjacent stops form a round. In other words, if the applicants have been on waiting lists for k rounds, they will give up and pursue other options.[5] We describe the k waiting lists by $(\mathbf{y}^{t,1}, \mathbf{y}^{t,2}, \ldots, \mathbf{y}^{t,k})$, where $\mathbf{y}^{t,l} = (y_1^{t,l}, y_2^{t,l}, \ldots, y_{n_t^l}^{t,l})$ represents the scores of the applicants who can be on the waiting list for at most l more rounds. For notational convenience, we let $\mathbf{y}^{t,k+1} = \mathbf{s}^t$ and $\mathbf{y}^t = (\mathbf{y}^{t,1}, \mathbf{y}^{t,2}, \ldots, \mathbf{y}^{t,k+1})$, and we simply call each $\mathbf{y}^{t,l}$ a list. The random variable $W_i^{t,l}$ represents whether applicant i who is on the lth list in period t departs in period $t+1$, where $W_i^{t,l} = 1$ if he or she departs in the next period and $W_i^{t,l} = 0$ otherwise. The departure probability of each applicant is $p \in (0, 1)$. We still use a_t to denote the stopping decision. If the recruiter waits ($a_t = 0$), more applicants will be available in the next period, while some on $\mathbf{y}^t$ may depart; if the recruiter stops ($a_t = 1$), they rank the n_t^l applicants on each $\mathbf{y}^{t,l}$ and then determine the number of offers to make, m_t^l, on each $\mathbf{y}^{t,l}$. Let $\mathbf{m}_t = (m_t^1, m_t^2, \ldots, m_t^{k+1})$ and $\mathbf{n}_t = (n_t^1, n_t^2, \ldots, n_t^{k+1})$.

The scores of the applicants on $\mathbf{y}^t$ transition from period t to $t+1$ as follows: $\mathbf{Y}^{t+1}(\mathbf{y}^t, \mathbf{m}_t, a_t) = (\mathbf{Y}^{t+1,1}, \mathbf{Y}^{t+2,1}, \ldots, \mathbf{Y}^{t+1,k+1})$, where for $1 \leq l \leq k$,

$$\mathbf{Y}^{t+1,l}(\mathbf{y}^t, \mathbf{m}_t, a_t) = \begin{cases} \hat{\mathbf{Y}}^{t+1}(\mathbf{y}^{t,l}) & \text{if } \ a_t = 0, \\ \hat{\mathbf{Y}}^{t+1}\left(y^{t,l+1}_{[m_t^{l+1}+1]}, y^{t,l+1}_{[m_t^{l+1}+2]}, \ldots, y^{t,l+1}_{[n_t^{l+1}]}\right) & \text{if } \ a_t = 1. \end{cases}$$

When $a_t = 0$, $\hat{\mathbf{Y}}^{t+1}(\mathbf{y}^{t,l}) = (y_i^{t,l})_{i \in \left\{j \in \left\{1,2,\ldots,n_t^l\right\}: W_j^{t,l}=0\right\}}$, which represents the scores of the applicants

[5]Another way to think about this is that applicants who have not received an offer after k rounds are most likely those who have very low scores, and the recruiter will have no need to recall them.

who are on the lth list in period t and have not departed in period $t+1$ if the recruiter waits. Similarly, when $a_t = 1$, $\hat{\mathbf{Y}}^{t+1}\left(y^{t,l+1}_{[m^{l+1}_t+1]}, y^{t,l+1}_{[m^{l+1}_t+2]}, \ldots, y^{t,l+1}_{[n^{l+1}_t]}\right) = (y^{t,l}_i)_{i\in\left\{j\in\Omega: W^{t,l+1}_j=0\right\}}$, which represents the scores of the applicants who are on the $(l+1)$th list and not hired in period t and have not departed in period $t+1$ if the recruiter stops. Here, Ω is the set of indices of the applicants on $\mathbf{y}^{t,l+1}$ who do not receive an offer in period t.

The scores of the applicants who have not been put on waiting lists have the following transitions: $\mathbf{Y}^{t+1,k+1}(\mathbf{y}^t, \mathbf{m}_t, a_t) = \mathbf{S}^{t+1}(\mathbf{y}^{t,k+1}, a_t)$. The dynamic programming formulation is as follows:

$$\hat{V}_t(q_t, \mathbf{y}^t) = \max\left\{\mathbb{E}\hat{V}_{t+1}(q_t, \mathbf{Y}^{t+1}(\mathbf{y}^t, \mathbf{0}, 0)), \max_{\mathbf{0}\le\mathbf{m}_t\le\mathbf{n}_t} \hat{J}_t(q_t, \mathbf{m}_t, \mathbf{y}^t)\right\},$$

where $\mathbf{0}$ is the zero vector and

$$\hat{J}_t(q_t, \mathbf{m}_t, \mathbf{y}^t) = \sum_{l=1}^{k+1}\sum_{i=1}^{m^l_t} y^{t,l}_{[i]} + \mathbb{E}\hat{V}_{t+1}\left(q_t + \sum_{l=1}^{k+1} m^l_t, \mathbf{Y}^{t+1}(\mathbf{y}^t, \mathbf{m}_t, 1)\right).$$

The boundary condition is $\hat{V}_{T+1}(q_{T+1}, \mathbf{y}^{T+1}) = -G(q_{T+1} - d)$. We let the optimal stopping rule $a^*_t(q_t, \mathbf{y}^t)$ be such that

$$\hat{V}_t(q_t, \mathbf{y}^t) = \begin{cases} \mathbb{E}\hat{V}_{t+1}(q_t, \mathbf{Y}^{t+1}(\mathbf{y}^t, \mathbf{0}, 0)) & \text{if } a^*_t(q_t, \mathbf{y}^t) = 0, \\ \max\limits_{\mathbf{0}\le\mathbf{m}_t\le\mathbf{n}_t} \hat{J}_t(q_t, \mathbf{m}_t, \mathbf{y}^t) & \text{if } a^*_t(q_t, \mathbf{y}^t) = 1, \end{cases}$$

and define the optimal number of offers to make as

$$\mathbf{m}^*_t(q_t, \mathbf{y}^t) = \arg\max_{\mathbf{0}\le\mathbf{m}_t\le\mathbf{n}_t} \hat{J}_t(q_t, \mathbf{m}_t, \mathbf{y}^t).$$

When there are multiple maximizers, $m^{l^*}_t(q_t, \mathbf{y}^t)$ is defined as the largest one. A simple sample-path argument can show that for each list $\mathbf{y}^{t,l}$, the optimal policy for issuing offers is of a cutoff type: an applicant receives an offer only if all of the more qualified applicants on the same list receive an offer. As a special case, when $k = 0$, the model above reduces to the base model discussed earlier. When $k \ge T$, the recruiter can recall any applicant who has not yet received an offer.

To keep track of the applicants on waiting lists, the state variables in the dynamic programs need to increase, which poses additional challenges in both analysis and computation. However, the main qualitative results that we showed earlier continue to hold.

Theorem 5.

(i) Theorems 1, 2, and 4 hold for applicants on $\mathbf{y}^{t,1}$.

(ii) For applicant j *on* $\mathbf{y}^{t,l}, l > 1$,

(1) there exists a threshold $U_j^{t,l} \in \mathbb{R}_+$ such that applicant j is hired if and only if $y_j^{t,l} \geq U_j^{t,l}$. Moreover, m_t^{l*} is constant for all $y_j^{t,l} \geq U_j^{t,l}$;

(2) for any $y_i^{t,l} > y_j^{t,l}$ and $\delta > 0$ such that $y_i^{t,l} - \delta \geq y_j^{t,l} + \delta$, the total reward of the recruiter under $\mathbf{y}^{t,l} + \delta \mathbf{e}_j - \delta \mathbf{e}_i$ is lower than that under $\mathbf{y}^{t,l}$.

(iii) For applicants i and j on $\mathbf{y}^{t,l'}$ and $\mathbf{y}^{t,l}$, respectively, with $l' < l$, if $y_i^{t,l'} \geq y_j^{t,l}$, then applicant i is hired if applicant j is hired.

Theorem 5(*i*) and (*ii*) show that the main qualitative results extend to the case with waiting lists and that these results are robust. What is crucial is that placing applicants on a waiting list will change their behavior. Part (iii) is new. Different lists are not treated the same way. The recruiter sets a lower threshold for an older list (on which applicants have been waiting longer) than for a more recent one because there is more urgency to make offers to those on the older list. This strategy is similar in spirit to clearing sales of perishable inventories, where older inventories should be cleared before younger ones (e.g., Li et al. 2016). A more general model of the waiting lists would assume a higher departure probability for those on the lists, though this would not qualitatively change the main results.

8 Concluding Remarks

In this paper, we explore the option of waiting in a rolling recruitment process where applicants depart stochastically. We characterize the optimal polices about the stopping rule and the number of offers made that maximize the recruiter's total reward over the recruitment season. Our findings show that when an applicant's score increases, the recruiter may change from stopping to waiting for a thicker market, and then stopping again when the score is sufficiently high. Applicants may be disadvantaged when another applicant's score increases, but they may also benefit from an increase in another applicant's score. Furthermore, the recruiter's reward is higher when the applicant scores in the current period or in future periods are more dispersed. Our numerical studies show that the value of waiting can be substantial when highly qualified applicants arrive infrequently, the hiring target is not too high, the arrival rate is low, or the departure rate is low. Our analytical analysis shows how such value can be realized through following the optimal policies or heuristics that are inspired by the optimal policies.

In our previous analysis, we made the assumption that all of the applicants available in a period depart the system in the next period with the same probability. Our main results hold when this assumption is relaxed. Another assumption that we made is that departures and scores are independent. Although this assumption is common in the literature (e.g., Akbarpour et al. 2020, Ashlagi et al. 2023, Kesselheim et al. 2024), in reality, departures depend on outside options, which in turn depend on applicant scores (i.e., qualifications). It would be a major undertaking to extend our model to incorporate the dependency between the departures and scores, and similar optimal policies hold only under some special cases of this dependency.

This study provides several other directions for future research. For example, it would be interesting to consider the possibility of applicants rejecting their offers in our model. Because incorporating such random yields into the rolling recruitment process substantially complicates the optimal policies, as shown in Du et al. (2024), approximations and heuristics, as opposed to structural properties, should be the focus. In our model, we assumed that applicants' true scores are known as soon as they arrive in a period. In practice, however, there might be bias in evaluations of applicants (Salem and Gupta 2023), or the true scores may be known only after applicants go through multiple costly tests (Du and Li 2020). In the latter case, the recruiter must determine how many applicants should be extended offers, how many should be rejected, and how many should proceed to the next test before an accept/reject decision is made.

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Appendix

Proof of Lemma 1: (i) The proof is by induction. $V_{T+1}(q_{T+1}, \mathbf{s}^{T+1})$ is obviously convex increasing in $\mathbf{s}^{T+1}$. Suppose that $V_{t+1}(q_{t+1}, \mathbf{s}^{t+1})$ is convex increasing in $\mathbf{s}^{t+1}$. We first show that

$$\max_{1\le m_t\le n_t} J_t(q_t, m_t, \mathbf{s}^t) = \max_{1\le m_t\le n_t} \left\{ \sum_{i=1}^{m_t} s^t_{[i]} + \mathbb{E}V_{t+1}(q_t+m_t, \mathbf{S}^{t+1}(\mathbf{s}^t, 1)) \right\}$$

is convex increasing in $\mathbf{s}^t$. Because

$$\sum_{i=1}^{m_t} s^t_{[i]} = \max\left\{ \sum_{k=1}^{m_t} s^t_{i_k} : 1 \le i_1 \le \cdots \le i_{m_t} \le n_t \right\}$$

is the maximum of a finite number of convex increasing functions of $\mathbf{s}^t$, it is convex increasing in $\mathbf{s}^t$. In addition, the state transition $\mathbf{S}^{t+1}(\mathbf{s}^t,1)=\mathbf{S}_B^{t+1}$ does not depend on $\mathbf{s}^t$, so $\mathbb{E}V_{t+1}(q_t+m_t,\mathbf{S}^{t+1}(\mathbf{s}^t,1))$ does not depend on $\mathbf{s}^t$. Therefore, $\max_{1\le m_t\le n_t} J_t(q_t,m_t,\mathbf{s}^t)$ is the maximum of n_t convex increasing functions of $\mathbf{s}^t$. So it is also convex increasing in $\mathbf{s}^t$.

Next, we show that $\mathbb{E}V_{t+1}(q_t,\mathbf{S}^{t+1}(\mathbf{s}^t,0))$ is convex increasing in $\mathbf{s}^t$. In this case, the state transition is $\mathbf{S}^{t+1}(\mathbf{s}^t,0)=(\mathbf{S}_A^{t+1}(\mathbf{s}^t),\mathbf{S}_B^{t+1})$. Because $V_{t+1}(q_t,\mathbf{s}^{t+1})$ is convex increasing in $\mathbf{s}^{t+1}$, for any realization $\mathbf{s}_B^{t+1}$ of $\mathbf{S}_B^{t+1}$ and $I\in 2^{\{1,2,\dots,n_t\}}$, $V_{t+1}(q_t,((s_i^t)_{i\in I},\mathbf{s}_B^{t+1}))$ is convex increasing in $\mathbf{s}^t$. Then,

$$
\begin{aligned}
&\mathbb{E}V_{t+1}(q_t,(\mathbf{S}_A^{t+1}(\mathbf{s}^t),\mathbf{s}_B^{t+1}))\\
=&\sum_{I\in 2^{\{1,2,\dots,n_t\}}} V_{t+1}(q_t,((s_i^t)_{i\in I},\mathbf{s}_B^{t+1}))\mathbb{P}(\mathbf{S}_A^{t+1}(\mathbf{s}^t)=(s_i^t)_{i\in I})\\
=&\sum_{I\in 2^{\{1,2,\dots,n_t\}}} V_{t+1}(q_t,((s_i^t)_{i\in I},\mathbf{s}_B^{t+1}))(1-p)^{|I|}p^{n_t-|I|}.
\end{aligned}
$$

Because $2^{\{1,2,\dots,n_t\}}$ is finite, $(1-p)^{|I|}p^{n_t-|I|}\ge 0$, and $V_{t+1}(q_t,((s_i^t)_{i\in I},\mathbf{s}_B^{t+1}))$ is convex increasing in $\mathbf{s}^t$ for each $I\in 2^{\{1,2,\dots,n_t\}}$, $\mathbb{E}V_{t+1}(q_t,(\mathbf{S}_A^{t+1}(\mathbf{s}^t),\mathbf{s}_B^{t+1}))$ is convex increasing in $\mathbf{s}^t$. Therefore,

$$
\mathbb{E}V_{t+1}(q_t,(\mathbf{S}_A^{t+1}(\mathbf{s}^t),\mathbf{S}_B^{t+1}))=\int_{\mathbf{S}_B^{t+1}}\mathbb{E}V_{t+1}(q_t,(\mathbf{S}_A^{t+1}(\mathbf{s}^t),\mathbf{s}_B^{t+1}))\,\mathrm{d}F_{\mathbf{S}_B^{t+1}}(\mathbf{s}_B^{t+1})
$$

is convex increasing in $\mathbf{s}^t$ because convexity and monotonicity are preserved under expectation. Here, $F_{\mathbf{S}_B^{t+1}}$ is the distribution function of $\mathbf{S}_B^{t+1}$, and the equality follows because $\mathbf{S}_B^{t+1}$ and $\mathbf{S}_A^{t+1}(\mathbf{s}^t)$ are independent and by Fubini's theorem.

Because $V_t(q_t,\mathbf{s}^t)$ is the maximization over two convex increasing functions in $\mathbf{s}^t$, it is also convex increasing in $\mathbf{s}^t$.

(ii) Let $\tilde{\mathbf{s}}^t=\mathbf{s}^t+\delta\mathbf{e}_j$. To prove $\nabla_{s_j^t}V_t(q_t,\mathbf{s}^t)\le 1$, it is equivalent to show that

$$
V_t(q_t,\tilde{\mathbf{s}}^t)-V_t(q_t,\mathbf{s}^t)\le\delta
$$

for any $\delta\ge 0$. We prove this through a sample-path argument.

Suppose that the optimal policy is π^* when the state is $(q_t,\tilde{\mathbf{s}}^t)$, and that we implement π^* when the state is $(q_t,\mathbf{s}^t)$, which may be suboptimal. The policy π^* specifies when to stop, how many offers to make, and whom to make offers to. Given a sample path, the difference between the total rewards under the two states is either zero (when applicant j is not hired under π^*) or δ (when applicant j is hired). Let $V_t^{\pi^*}(q_t,\mathbf{s}^t)$ be the total reward when the state is $(q_t,\mathbf{s}^t)$ and π^* is implemented. Then,

$$
\begin{aligned}
V_t(q_t,\tilde{\mathbf{s}}^t)-V_t(q_t,\mathbf{s}^t)\le& V_t(q_t,\tilde{\mathbf{s}}^t)-V_t^{\pi^*}(q_t,\mathbf{s}^t)\\
\le&\delta,
\end{aligned}
$$

where the first inequality follows because π^* is feasible when the state is $(q_t, \mathbf{s}^t)$.

(iii) If applicant j departs in period $t+1$, then the marginal value of j's score in period $t+1$ is obviously zero. Therefore,

$$
\begin{aligned}
&\nabla_{s_j^t}\mathbb{E}V_{t+1}(q_t, \mathbf{S}^{t+1}(\mathbf{s}^t, 0))\\
=&(1-p)\nabla_{s_j^t}\mathbb{E}\left[V_{t+1}(q_t, \mathbf{S}^{t+1}(\mathbf{s}^t, 0))\middle|W_j^t = 0\right] + p\nabla_{s_j^t}\mathbb{E}\left[V_{t+1}(q_t, \mathbf{S}^{t+1}(\mathbf{s}^t, 0))\middle|W_j^t = 1\right]\\
=&(1-p)\mathbb{E}\left[\nabla_{s_j^t}V_{t+1}(q_t, \mathbf{S}^{t+1}(\mathbf{s}^t, 0))\middle|W_j^t = 0\right]\\
\le&1-p,
\end{aligned}
$$

where the inequality is by (ii). ■

Proof of Lemma 2: For notational brevity, let $\tilde{\mathbf{s}} = (s_1, \ldots, s_{j-1}, \tilde{s}_j, s_{j+1}, \ldots, s_n)$ and $\mathbf{s}' = (s_1, \ldots, s_{j-1}, s'_j, s_{j+1}, \ldots, s_n)$.

(i) We first show that $\{s_j \ge 0 : s_j \ge s_{[m^*(\mathbf{s})]}\}$ is nonempty and that its minimum is attainable. Here, $s_{[m^*(\mathbf{s})]}$ as a function of s_j may not be monotonic and continuous. Let $s_j = \max_{i\in\{1,2,\ldots,n\}\setminus\{j\}} s_i$. Because $s_j = s_{[1]} \ge s_{[m^*(\mathbf{s})]}$, the set $\{s_j \ge 0 : s_j \ge s_{[m^*(\mathbf{s})]}\}$ is nonempty. Define $c_j(\mathbf{s}_{-j}) = \inf\{s_j \ge 0 : s_j \ge s_{[m^*(\mathbf{s})]}\}$ and let $\mathbf{s}^c = (s_1, \ldots, s_{j-1}, c_j(\mathbf{s}_{-j}), s_{j+1}, \ldots, s_n)$. To show that the infimum can be replaced by the minimum, it suffices to show that $c_j(\mathbf{s}_{-j}) \ge s^c_{[m^*(\mathbf{s}^c)]}$.

Take any $\varepsilon > 0$ and let $\tilde{s}_j = c_j(\mathbf{s}_{-j}) + \varepsilon$. We first show that $j \in M(\tilde{\mathbf{s}})$ by contradiction. Suppose $j \notin M(\tilde{\mathbf{s}})$. By the definition of $c_j(\mathbf{s}_{-j})$, there exists an $s'_j \in \{s_j \ge 0 : s_j \ge s_{[m^*(\mathbf{s})]}\}$ such that $s'_j < c_j(\mathbf{s}_{-j}) + \varepsilon = \tilde{s}_j$. This implies that $s'_{[m^*(\mathbf{s}')]} \le s'_j < \tilde{s}_j < \tilde{s}_{[m^*(\tilde{\mathbf{s}})]}$, and therefore, $m^*(\mathbf{s}') > m^*(\tilde{\mathbf{s}})$. Because

$$\sum_{i=1}^{m} s_{[i]} + g(m) - \left(\sum_{i=1}^{m-1} s_{[i]} + g(m-1)\right) = s_{[m]} + g(m) - g(m-1)$$

is increasing in s_j, $\sum_{i=1}^{m} s_{[i]} + g(m)$ is supermodular in (m, s_j), which furhter implies that $m^*(\mathbf{s})$ is increasing in s_j. Thus, we have $m^*(\mathbf{s}') \le m^*(\tilde{\mathbf{s}})$, which is a contradiction. Therefore, $j \in M(\tilde{\mathbf{s}})$. Then, we have $\tilde{s}_j \ge \tilde{s}_{[m^*(\tilde{\mathbf{s}})]} \ge s^c_{[m^*(\tilde{\mathbf{s}})]}$, where the second inequality holds because $\tilde{\mathbf{s}}$ only differs from $\mathbf{s}^c$ in the jth component and $\tilde{s}_j > s^c_j$. Because $m^*(\mathbf{s})$ is the largest maximizer in (3), it is clear that $m^*(\mathbf{s})$ is a step function of s_j and is right continuous in s_j. Therefore,

$$c_j(\mathbf{s}_{-j}) = \lim_{\varepsilon\to 0+}(c_j(\mathbf{s}_{-j}) + \varepsilon) = \lim_{\varepsilon\to 0+}\tilde{s}_j \ge \lim_{\varepsilon\to 0+} s^c_{[m^*(\tilde{\mathbf{s}})]} = s^c_{[m^*(\mathbf{s}^c)]}.$$

Hence, $c_j(\mathbf{s}_{-j}) \ge s^c_{[m^*(\mathbf{s}^c)]}$, and the minimum of $\{s_j \ge 0 : s_j \ge s_{[m^*(\mathbf{s})]}\}$ is attainable.

To prove the second and third statements of (i), we first prove the following claim: $m^*(\tilde{\mathbf{s}}) = m^*(\mathbf{s})$ for any $\tilde{s}_j \ge s_j$ with $j \in M(\mathbf{s})$ and for any $\tilde{s}_j \le s_j$ with $j \notin M(\mathbf{s})$.

For any $m^*(\mathbf{s}) < m' \le n$, by the optimality of $m^*(\mathbf{s})$, we have

$$
\begin{aligned}
0 >& \sum_{i=1}^{m'} s_{[i]} + g(m') - \left(\sum_{i=1}^{m^*(\mathbf{s})} s_{[i]} + g(m^*(\mathbf{s})) \right) \\
=& \sum_{i=m^*(\mathbf{s})+1}^{m'} s_{[i]} + g(m') - g(m^*(\mathbf{s})).
\end{aligned} \tag{A1}
$$

Similarly, for any $1 \le m'' < m^*(\mathbf{s})$, we have

$$
\begin{aligned}
0 \ge& \sum_{i=1}^{m''} s_{[i]} + g(m'') - \left(\sum_{i=1}^{m^*(\mathbf{s})} s_{[i]} + g(m^*(\mathbf{s})) \right) \\
=& - \sum_{i=m''+1}^{m^*(\mathbf{s})} s_{[i]} + g(m'') - g(m^*(\mathbf{s})).
\end{aligned} \tag{A2}
$$

We consider the following two cases. In both cases, we show that $m^*(\tilde{\mathbf{s}}) = m^*(\mathbf{s})$ by contradiction.

Case 1. Take any $\tilde{s}_j > s_j$ with $j \in M(\mathbf{s})$. Then, $m^*(\tilde{\mathbf{s}}) \ge m^*(\mathbf{s})$. Suppose $m^*(\tilde{\mathbf{s}}) > m^*(\mathbf{s})$. Then,

$$
\begin{aligned}
f(\tilde{\mathbf{s}}) &= \sum_{i=1}^{m^*(\tilde{\mathbf{s}})} \tilde{s}_{[i]} + g(m^*(\tilde{\mathbf{s}})) \\
&= \sum_{i=1}^{m^*(\mathbf{s})} \tilde{s}_{[i]} + \sum_{i=m^*(\mathbf{s})+1}^{m^*(\tilde{\mathbf{s}})} \tilde{s}_{[i]} + g(m^*(\tilde{\mathbf{s}})) \\
&= \sum_{i=1}^{m^*(\mathbf{s})} \tilde{s}_{[i]} + \sum_{i=m^*(\mathbf{s})+1}^{m^*(\tilde{\mathbf{s}})} s_{[i]} + g(m^*(\tilde{\mathbf{s}})) \\
&= \sum_{i=1}^{m^*(\mathbf{s})} \tilde{s}_{[i]} + g(m^*(\mathbf{s})) + \sum_{i=m^*(\mathbf{s})+1}^{m^*(\tilde{\mathbf{s}})} s_{[i]} + g(m^*(\tilde{\mathbf{s}})) - g(m^*(\mathbf{s})) \\
&< \sum_{i=1}^{m^*(\mathbf{s})} \tilde{s}_{[i]} + g(m^*(\mathbf{s})),
\end{aligned}
$$

which contradicts the optimality of $m^*(\tilde{\mathbf{s}})$. Here, the third equality holds because $\tilde{s}_{[i]} = s_{[i]}$ for all $i \ge m^*(\mathbf{s}) + 1$, and the inequality is by Equation (A1). Hence, we have $m^*(\tilde{\mathbf{s}}) = m^*(\mathbf{s})$.

In this case, $f(\mathbf{s})$ can be written as

$$
f(\mathbf{s}) = s_j + \left(\sum_{i=1}^{m^*(\mathbf{s})} s_{[i]} - s_j \right) + g(m^*(\mathbf{s})), \tag{A3}
$$

where $\sum_{i=1}^{m^*(\mathbf{s})} s_{[i]} - s_j$ represents the total score of the highest $m^*(\mathbf{s})$ scores in $\mathbf{s}$ except for s_j. Because $s_j \ge s_{[m^*(\mathbf{s})]}$, $\tilde{s}_j \ge \tilde{s}_{[m^*(\mathbf{s})]}$, and $m^*(\tilde{\mathbf{s}}) = m^*(\mathbf{s})$, we have $\sum_{i=1}^{m^*(\tilde{\mathbf{s}})} \tilde{s}_{[i]} - \tilde{s}_j = \sum_{i=1}^{m^*(\mathbf{s})} s_{[i]} - s_j$, and therefore,

$$\begin{aligned}
f(\tilde{\mathbf{s}}) = &\sum_{i=1}^{m^*(\tilde{\mathbf{s}})} \tilde{s}_{[i]} + g(m^*(\tilde{\mathbf{s}})) \\
=&\tilde{s}_j + \left(\sum_{i=1}^{m^*(\mathbf{s})} s_{[i]} - s_j \right) + g(m^*(\mathbf{s})) \\
=&\tilde{s}_j - s_j + f(\mathbf{s}),
\end{aligned} \tag{A4}$$

where the last equality holds by Equation (A3).

Case 2. Take any $\tilde{s}_j < s_j$ with $j \notin M(\mathbf{s})$. Then, $m^*(\tilde{\mathbf{s}}) \leq m^*(\mathbf{s})$. Suppose $m^*(\tilde{\mathbf{s}}) < m^*(\mathbf{s})$. Then,

$$\begin{aligned}
f(\tilde{\mathbf{s}}) &= \sum_{i=1}^{m^*(\tilde{\mathbf{s}})} \tilde{s}_{[i]} + g(m^*(\tilde{\mathbf{s}})) \\
&= \sum_{i=1}^{m^*(\tilde{\mathbf{s}})} s_{[i]} + g(m^*(\tilde{\mathbf{s}})) \\
&= \sum_{i=1}^{m^*(\mathbf{s})} s_{[i]} - \sum_{i=m^*(\tilde{\mathbf{s}})+1}^{m^*(\mathbf{s})} s_{[i]} + g(m^*(\tilde{\mathbf{s}})) \\
&= \sum_{i=1}^{m^*(\mathbf{s})} s_{[i]} + g(m^*(\mathbf{s})) - \sum_{i=m^*(\tilde{\mathbf{s}})+1}^{m^*(\mathbf{s})} s_{[i]} + g(m^*(\tilde{\mathbf{s}})) - g(m^*(\mathbf{s})) \\
&\leq \sum_{i=1}^{m^*(\mathbf{s})} s_{[i]} + g(m^*(\mathbf{s})) \\
&= \sum_{i=1}^{m^*(\mathbf{s})} \tilde{s}_{[i]} + g(m^*(\mathbf{s})),
\end{aligned}$$

which contradicts the optimality of $m^*(\tilde{\mathbf{s}})$. Here, the second equality holds because $\tilde{s}_{[i]} = s_{[i]}$ for all $i \leq m^*(\mathbf{s})$, and the inequality is by Equation (A2). Hence, we have $m^*(\tilde{\mathbf{s}}) = m^*(\mathbf{s})$.

In this case, we have

$$\begin{aligned}
f(\tilde{\mathbf{s}}) &= \sum_{i=1}^{m^*(\tilde{\mathbf{s}})} \tilde{s}_{[i]} + g(m^*(\tilde{\mathbf{s}})) \\
&= \sum_{i=1}^{m^*(\mathbf{s})} s_{[i]} + g(m^*(\mathbf{s})) \\
&= f(\mathbf{s}).
\end{aligned} \tag{A5}$$

Then, the claim follows by Cases 1 and 2.

Now, let $s_j = c_j(\mathbf{s}_{-j})$. We prove the second statement of (i). By the definition of $c_j(\mathbf{s}_{-j})$, we have $s_j \geq s_{[m^*(\mathbf{s})]}$, i.e., $j \in M(\mathbf{s})$. Then, by the claim, $m^*(\tilde{\mathbf{s}}) = m^*(\mathbf{s})$ for any $\tilde{s}_j \geq s_j$. In this case, we let $\overline{m} = m^*(\mathbf{s})$. Similarly, we have $\tilde{s}_j < \tilde{s}_{[m^*(\tilde{\mathbf{s}})]}$ for any $\tilde{s}_j < s_j$, i.e., $j \notin M(\tilde{\mathbf{s}})$. Then, by the claim, $m^*(\mathbf{s}') = m^*(\tilde{\mathbf{s}})$ for any $s'_j \leq \tilde{s}_j$. Because $\tilde{s}_j$ is arbitrary, $m^*(\tilde{\mathbf{s}})$ is constant for all $\tilde{s}_j < s_j$, and we denote the constant by $\underline{m}$. Finally, because $m^*(\mathbf{s})$ is increasing in $\mathbf{s}$, we have $\underline{m} \leq \overline{m}$.

For the third statement of (i), the necessity part holds by the definition of $c_j(\mathbf{s}_{-j})$. For the sufficiency part, because $m^*(\tilde{\mathbf{s}}) = m^*(\mathbf{s})$ for any $\tilde{s}_j \geq s_j$ when $s_j = c_j(\mathbf{s}_{-j})$, it follows that $j \in M(\tilde{\mathbf{s}})$.

(ii) Let $s_j = c_j(\mathbf{s}_{-j})$. Then, $j \in M(\mathbf{s})$ by (i). Hence, $f(\tilde{\mathbf{s}}) = \tilde{s}_j - s_j + f(\mathbf{s})$ for all $\tilde{s}_j \geq s_j$ by Equation (A4). Take any $s'_j < s_j$. Then, $j \notin M(\mathbf{s}')$ by (i). Hence, $f(\tilde{\mathbf{s}}) = f(\mathbf{s}')$ for all $\tilde{s}_j \leq s'_j$ by Equation (A5). Note that $f(\cdot)$ is continuous by its convexity. Then, letting $s'_j \to s_j-$, we have $f(\tilde{\mathbf{s}}) = f(\mathbf{s})$ for all $\tilde{s}_j < s_j$. Therefore, we have

$$f(\tilde{\mathbf{s}}) = (\tilde{s}_j - s_j)^+ + f(\mathbf{s}).$$

Applying the above equation for $\tilde{s}_j = 0$, we have $f(s_1, \ldots, s_{j-1}, 0, s_{j+1}, \ldots, s_n) = (0 - s_j)^+ + f(\mathbf{s}) = f(\mathbf{s})$. ■

Proof of Theorem 1: (i) To simplify notation, we suppress q_t and $\mathbf{s}^t_{-j}$ and define

$$f(s^t_j) = \max_{1 \leq m_t \leq n_t} \left\{ \sum_{i=1}^{m_t} s^t_{[i]} + \mathbb{E}V_{t+1}(q_t + m_t, \mathbf{S}^{t+1}(\mathbf{s}^t, 1)) \right\}, \tag{A6}$$

and

$$g(s^t_j) = \mathbb{E}V_{t+1}(q_t, \mathbf{S}^{t+1}(\mathbf{s}^t, 0)). \tag{A7}$$

Note that $f(s^t_j)$ and $g(s^t_j)$ are continuously increasing in s^t_j. By Lemma 2(i), we can define the value $c^t_j = \min \left\{ s^t_j \geq 0 : s^t_j \geq s^t_{[m^*_t(q_t, \mathbf{s}^t)]} \right\}$. By Lemma 2$(ii)$, $f(s^t_j)$ can be written as

$$f(s^t_j) = (s^t_j - c^t_j)^+ + f(0).$$

Therefore, $\nabla_{s^t_j} f(s^t_j) = 0$ on $[0, c^t_j)$ and $\nabla_{s^t_j} f(s^t_j) = 1$ on $[c^t_j, \infty)$. In addition, we have $\nabla_{s^t_j} g(s^t_j) \leq 1 - p \leq 1$ on $[0, \infty)$ by Lemma 1(iii). Because when $p = 0$, it is optimal for the recruiter to wait until the end, we let $L^t_j(q_t, \mathbf{s}^t_{-j}) = -\infty$ and $U^t_j(q_t, \mathbf{s}^t_{-j}) = \infty$ for all $t < T$, and focus on the cases when $p \in (0, 1]$ or when $p = 0$ and $t = T$. We consider the following two cases.

Case 1. If $g(c^t_j) \leq f(c^t_j)$, then $g(s^t_j) \leq f(s^t_j)$ for all $s^t_j \geq 0$. Let

$$L^t_j(q_t, \mathbf{s}^t_{-j}) = U^t_j(q_t, \mathbf{s}^t_{-j}) = c^t_j. \tag{A8}$$

Then, we have $a^*_t(q_t, \mathbf{s}^t) = 1$ for all $s^t_j \geq 0$.

Case 2. If $g(c^t_j) > f(c^t_j)$, then there exists a $\tilde{s}^t_j > c^t_j$ such that $g(\tilde{s}^t_j) = f(\tilde{s}^t_j)$, $g(s^t_j) > f(s^t_j)$ for all $s^t_j \in (c^t_j, \tilde{s}^t_j)$ and $g(s^t_j) < f(s^t_j)$ for all $s^t_j \in (\tilde{s}^t_j, \infty)$. Let

$$U^t_j(q_t, \mathbf{s}^t_{-j}) = \tilde{s}^t_j, \tag{A9}$$

and

$$L_j^t(q_t, \mathbf{s}_{-j}^t) = \begin{cases} \inf\left\{s_j^t \in [0, c_j^t] : g(s_i^t) > f(s_i^t)\right\} & \text{if } g(0) \le f(0), \\ -\infty & \text{if } g(0) > f(0). \end{cases} \tag{A10}$$

Then, we have $g(s_j^t) > f(s_j^t)$ ($a_t^*(q_t, \mathbf{s}^t) = 0$) if $L_j^t(q_t, \mathbf{s}_{-j}^t) < s_j^t < U_j^t(q_t, \mathbf{s}_{-j}^t)$ and $g(s_i^t) \le f(s_i^t)$ ($a_t^*(q_t, \mathbf{s}^t) = 1$) if $s_j^t \le L_j^t(q_t, \mathbf{s}_{-j}^t)$ or $s_j^t \ge U_j^t(q_t, \mathbf{s}_{-j}^t)$.

(ii) By (i), when $s_j^t \le L_j^t(q_t, \mathbf{s}_{-j}^t)$ or $s_j^t \ge U_j^t(q_t, \mathbf{s}_{-j}^t)$, we have

$$V_t(q_t, \mathbf{s}^t) = \max_{1 \le m_t \le n_t} \left\{ \sum_{i=1}^{m_t} s_{[i]}^t + \mathbb{E}V_{t+1}(q_t + m_t, \mathbf{S}^{t+1}(\mathbf{s}^t, 1)) \right\}.$$

From Case 1 and Case 2 above, we have $L_j^t(q_t, \mathbf{s}_{-j}^t) \le c_j^t \le U_j^t(q_t, \mathbf{s}_{-j}^t)$. Then, applying Lemma 2$(i)$, we obtain the result.

(iii) We first prove the necessity part by contradiction. Suppose applicant j is hired but $s_j^t < U_j^t(q_t, \mathbf{s}_{-j}^t)$. If $L_j^t(q_t, \mathbf{s}_{-j}^t) < s_j^t < U_j^t(q_t, \mathbf{s}_{-j}^t)$, then j cannot be hired because $a_t^*(q_t, \mathbf{s}^t) = 0$. If $s_j^t \le L_j^t(q_t, \mathbf{s}_{-j}^t)$, then $s_j^t < c_j^t$ because $s_j^t < U_j^t(q_t, \mathbf{s}_{-j}^t)$. By the definition of c_j^t, we have $s_j^t < s_{[m_t^*(q_t, \mathbf{s}^t)]}^t$, so j is not hired. Hence, we must have $s_j^t \ge U_j^t(q_t, \mathbf{s}_{-j}^t)$.

For the sufficiency part, if $s_j^t \ge U_j^t(q_t, \mathbf{s}_{-j}^t) \ge c_j^t$, then $s_j^t \ge s_{[m^*(\mathbf{s}^t)]}^t$ by Lemma 2(i). If $s_j^t > s_{[m_t^*(q_t, \mathbf{s}^t)]}^t$, then applicant j is hired; if $s_j^t = s_{[m^*(\mathbf{s}^t)]}^t$, hiring applicant j is always optimal. ■

Proof of Lemma 3: (i) Note that $\mathbf{s} = \hat{\mathbf{s}}$ when $s_j = 0$ and $\mathbf{s} = \check{\mathbf{s}}$ when $s_j = c_i(\hat{\mathbf{s}}_{-i})$. In this proof, we use these two points and Lemma 2(ii) to determine $f(\mathbf{s})$ as a function of s_j and s_i.

By Lemma 2(ii), we have

$$f(\check{\mathbf{s}}) = (s_i - c_i(\check{\mathbf{s}}_{-i}))^+ + f(\mathbf{s}'), \tag{A11}$$

where $\mathbf{s}' = (s_1, \ldots, s_{j-1}, c_i(\hat{\mathbf{s}}_{-i}), s_{j+1}, \ldots, s_{i-1}, 0, s_{i+1}, \ldots, s_n)$. Let

$$\mathbf{s}'' = (s_1, \ldots, s_{j-1}, s_j, s_{j+1}, \ldots, s_{i-1}, 0, s_{i+1}, \ldots, s_n).$$

Then, $c_j(\mathbf{s}''_{-j}) = c_i(\hat{\mathbf{s}}_{-i})$ because interchanging values in the ith and jth coordinates does not change f. Then, by Lemma 2(ii),

$$\begin{aligned} f(\hat{\mathbf{s}}) =& (s_i - c_i(\hat{\mathbf{s}}_{-i}))^+ + f(s_1, \ldots, s_{j-1}, 0, s_{j+1}, \ldots, s_{i-1}, 0, s_{i+1}, \ldots, s_n) \\ =& (s_i - c_i(\hat{\mathbf{s}}_{-i}))^+ + \left[(c_i(\hat{\mathbf{s}}_{-i}) - c_j(\mathbf{s}''_{-j}))^+ + f(s_1, \ldots, s_{j-1}, 0, s_{j+1}, \ldots, s_{i-1}, 0, s_{i+1}, \ldots, s_n)\right] \\ =& (s_i - c_i(\hat{\mathbf{s}}_{-i}))^+ + f(s_1, \ldots, s_{j-1}, c_i(\hat{\mathbf{s}}_{-i}), s_{j+1}, \ldots, s_{i-1}, 0, s_{i+1}, \ldots, s_n) \\ =& (s_i - c_i(\hat{\mathbf{s}}_{-i}))^+ + f(\mathbf{s}'), \end{aligned} \tag{A12}$$

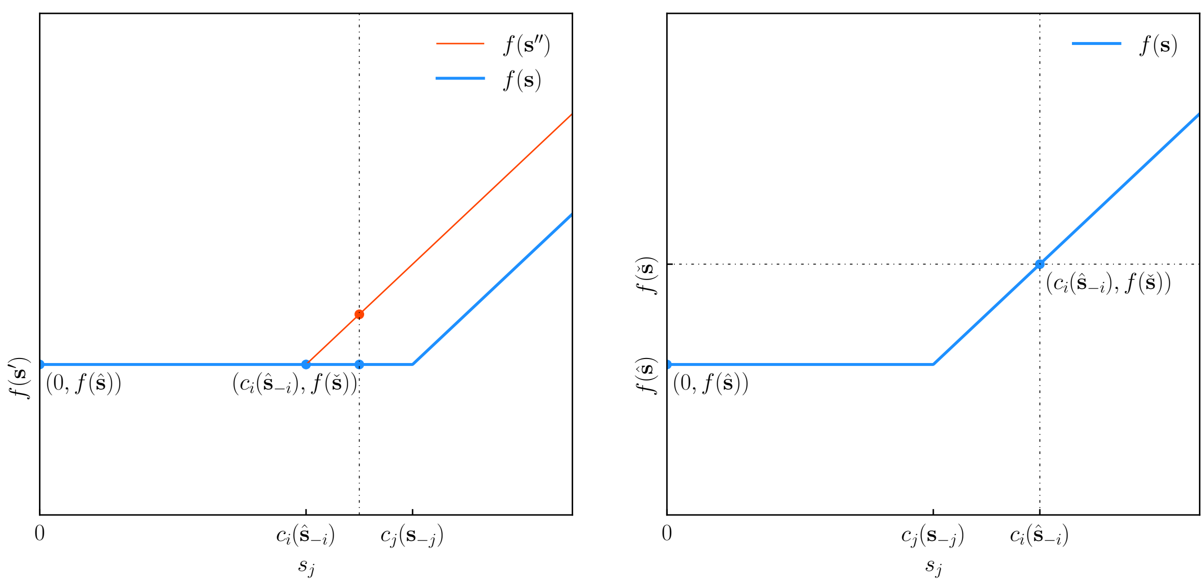
where the third equality follows by applying Lemma 2(ii) to $f(\mathbf{s}'')$ with respect to s_j and letting $s_j = c_i(\hat{\mathbf{s}}_{-i})$.

Next, we show that $c_i(\hat{\mathbf{s}}_{-i}) \geq c_i(\check{\mathbf{s}}_{-i})$ by contradiction. Suppose $c_i(\hat{\mathbf{s}}_{-i}) < c_i(\check{\mathbf{s}}_{-i})$. Take any $s_i \in (c_i(\hat{\mathbf{s}}_{-i}), c_i(\check{\mathbf{s}}_{-i}))$. Then, by Equations (A11) and (A12), $f(\check{\mathbf{s}}) = f(\mathbf{s}')$ and $f(\hat{\mathbf{s}}) = s_i - c_i(\hat{\mathbf{s}}_{-i}) + f(\mathbf{s}') > f(\check{\mathbf{s}})$, which is a contradiction because $f(\cdot)$ is increasing. Hence, we have $c_i(\hat{\mathbf{s}}_{-i}) \geq c_i(\check{\mathbf{s}}_{-i})$. We now consider the following three cases, depending on the value of s_i.

Case 1. $c_i(\hat{\mathbf{s}}_{-i}) > c_i(\check{\mathbf{s}}_{-i})$. In this case, we consider the following two subcases.

Figure A1: (Color online) Graphical Illustrations for Case 1

(a) Subcase 1 (b) Subcase 2



Subcase 1. $s_i \leq c_i(\check{\mathbf{s}}_{-i})$. In this subcase, we show $c_j(\mathbf{s}_{-j}) = c_i(\hat{\mathbf{s}}_{-i})$ by contradiction. By Equations (A11) and (A12), $f(\hat{\mathbf{s}}) = f(\check{\mathbf{s}}) = f(\mathbf{s}')$. This implies that $f(\mathbf{s}) = f(\mathbf{s}')$ for all $s_j \leq c_i(\hat{\mathbf{s}}_{-i})$, and therefore, $c_j(\mathbf{s}_{-j}) \geq c_i(\hat{\mathbf{s}}_{-i})$. Because $f(\mathbf{s}) \geq f(\mathbf{s}'')$ by monotonicity, we must have $c_j(\mathbf{s}_{-j}) \leq c_j(\mathbf{s}''_{-j}) = c_i(\hat{\mathbf{s}}_{-i})$. Otherwise, $f(\mathbf{s}) < f(\mathbf{s}'')$ for any $s_j > c_i(\hat{\mathbf{s}}_{-i})$, which is a contradiction (see Figure A1(a)). Therefore, we have $c_j(\mathbf{s}_{-j}) = c_i(\hat{\mathbf{s}}_{-i})$. Then, by Lemma 2($ii$), for all $s_j \geq 0$ and $s_i \leq c_i(\check{\mathbf{s}}_{-i})$,

$$
\begin{aligned}
f(\mathbf{s}) =& (s_j - c_j(\mathbf{s}_{-j}))^+ + f(\hat{\mathbf{s}}) \\
=& (s_j - c_i(\hat{\mathbf{s}}_{-i}))^+ + f(\hat{\mathbf{s}}).
\end{aligned}
$$

Subcase 2. $s_i > c_i(\check{\mathbf{s}}_{-i})$. By Equations (A11) and (A12), $f(\hat{\mathbf{s}}) = (s_i - c_i(\hat{\mathbf{s}}_{-i}))^+ + f(\mathbf{s}') < s_i - c_i(\check{\mathbf{s}}_{-i}) + f(\mathbf{s}') = f(\check{\mathbf{s}})$. Then, using the points $(0, f(\hat{\mathbf{s}}))$ and $(c_i(\hat{\mathbf{s}}_{-i}), f(\check{\mathbf{s}}))$ and applying Lemma

2(ii), we have (see Figure A1(b))

$$\begin{aligned}c_j(\mathbf{s}_{-j}) =& c_i(\hat{\mathbf{s}}_{-i}) - (f(\check{\mathbf{s}}) - f(\hat{\mathbf{s}}))\\ =& c_i(\hat{\mathbf{s}}_{-i}) - (s_i - c_i(\check{\mathbf{s}}_{-i})) + (s_i - c_i(\hat{\mathbf{s}}_{-i}))^+\\ =& c_i(\hat{\mathbf{s}}_{-i}) - \min\{c_i(\hat{\mathbf{s}}_{-i}), s_i\} + c_i(\check{\mathbf{s}}_{-i}).\end{aligned}$$

Applying Lemma 2(ii) again, for all $s_j \geq 0$ and $s_i > c_i(\check{\mathbf{s}}_{-i})$,

$$\begin{aligned}f(\mathbf{s}) &= (s_j - c_j(\mathbf{s}_{-j}))^+ + f(\hat{\mathbf{s}})\\ &= (s_j - (c_i(\hat{\mathbf{s}}_{-i}) - \min\{c_i(\hat{\mathbf{s}}_{-i}), s_i\} + c_i(\check{\mathbf{s}}_{-i})))^+ + f(\hat{\mathbf{s}}).\end{aligned}$$

By Subcases 1 and 2, for all $s_j \geq 0$ and $s_i \geq 0$,

$$f(\mathbf{s}) = (s_j - (c_i(\hat{\mathbf{s}}_{-i}) - \min\{c_i(\hat{\mathbf{s}}_{-i}), s_i\} + \min\{c_i(\check{\mathbf{s}}_{-i}), s_i\}))^+ + f(\hat{\mathbf{s}}). \tag{A13}$$

For Cases 2 and 3, let $\bar{\mathbf{s}} = (s_1, \ldots, s_{j-1}, c_i(\hat{\mathbf{s}}_{-i}), s_{j+1}, \ldots, s_{i-1}, c_i(\check{\mathbf{s}}_{-i}), s_{i+1}, \ldots, s_n)$ and define $b = \bar{s}_{[m^*(\bar{\mathbf{s}})-1]}$ if $m^*(\bar{\mathbf{s}}) > 1$ or $b = \infty$ otherwise.

Case 2. $c_i(\hat{\mathbf{s}}_{-i}) = c_i(\check{\mathbf{s}}_{-i}) < b$. In this case, we have $f(\mathbf{s}) = (s_j - c_i(\hat{\mathbf{s}}_{-i}))^+ + f(\hat{\mathbf{s}})$ for all $s_j \geq 0$ and $s_i \leq c_i(\check{\mathbf{s}}_{-i}) = c_i(\hat{\mathbf{s}}_{-i})$ by applying the same argument in Subcase 1 of Case 1. If $s_i > c_i(\check{\mathbf{s}}_{-i}) = c_i(\hat{\mathbf{s}}_{-i})$, we show that $\bar{s}_{[m^*(\bar{\mathbf{s}})]} = c_i(\hat{\mathbf{s}}_{-i})$.

By Lemma 2(i), $\check{s}_i = c_i(\check{\mathbf{s}}_{-i})$ implies that $\check{s}_i \geq \check{s}_{[m^*(\check{\mathbf{s}})]}$. Then, by $c_i(\check{\mathbf{s}}_{-i}) = c_i(\hat{\mathbf{s}}_{-i})$ and the construction of $\bar{\mathbf{s}}$ and $\check{\mathbf{s}}$, we have $\bar{s}_j \geq \bar{s}_{[m^*(\bar{\mathbf{s}})]}$ as well. This implies that $\bar{s}_j = \bar{s}_{[m^*(\bar{\mathbf{s}})]}$ because $\bar{s}_j = c_i(\hat{\mathbf{s}}_{-i}) < b$. Therefore, $\bar{s}_{[m^*(\bar{\mathbf{s}})]} = c_i(\hat{\mathbf{s}}_{-i})$.

Then, for any $s_i > c_i(\check{\mathbf{s}}_{-i}) = c_i(\hat{\mathbf{s}}_{-i})$, we have

$$\begin{aligned}0 >& \max_{1\leq k\leq n-m^*(\bar{\mathbf{s}})}\left\{\sum_{l\neq m^*(\bar{\mathbf{s}})}^{m^*(\bar{\mathbf{s}})+k} \bar{s}_{[l]} + s_i + g(m^*(\bar{\mathbf{s}})+k)\right\} - \left(\sum_{i=1}^{m^*(\bar{\mathbf{s}})-1} \bar{s}_{[i]} + s_i + g(m^*(\bar{\mathbf{s}}))\right)\\ =& \max_{1\leq k\leq n-m^*(\bar{\mathbf{s}})}\left\{\sum_{\substack{l\neq m^*(\bar{\mathbf{s}}),\\ l\neq m^*(\bar{\mathbf{s}})+1}}^{m^*(\bar{\mathbf{s}})+k} \bar{s}_{[l]} + s_i + c_i(\hat{\mathbf{s}}_{-i}) + g(m^*(\bar{\mathbf{s}})+k)\right\} - \left(\sum_{i=1}^{m^*(\bar{\mathbf{s}})-1} \bar{s}_{[i]} + s_i + g(m^*(\bar{\mathbf{s}}))\right)\\ =& c_i(\hat{\mathbf{s}}_{-i}) - b',\end{aligned}$$

where $b' = g(m^*(\bar{\mathbf{s}})) - \max_{1\leq k\leq n-m^*(\bar{\mathbf{s}})}\left\{\sum_{l=m^*(\bar{\mathbf{s}})+2}^{m^*(\bar{\mathbf{s}})+k} \bar{s}_{[l]} + g(m^*(\bar{\mathbf{s}})+k)\right\}$, and $j \in M(\mathbf{s})$ if and only if either one of the following holds: (1) $s_j < \min\{b, s_i\}$ and both i and j are on the offer list, i.e.,

$$\begin{aligned}0 \leq& \max_{1\leq k\leq n-m^*(\bar{\mathbf{s}})}\left\{\sum_{\substack{l\neq m^*(\bar{\mathbf{s}}),\\ l\neq m^*(\bar{\mathbf{s}})+1}}^{m^*(\bar{\mathbf{s}})+k} \bar{s}_{[l]} + s_i + s_j + g(m^*(\bar{\mathbf{s}})+k)\right\} - \left(\sum_{l=1}^{m^*(\bar{\mathbf{s}})-1} \bar{s}_{[l]} + s_i + g(m^*(\bar{\mathbf{s}}))\right)\\ =& s_j - b';\end{aligned}$$

or (2) $s_j \geq \min\{b, s_i\}$.

Let $\hat{b} = \min\{b, b'\}$. Clearly, $c_i(\hat{\mathbf{s}}_{-i}) < \hat{b} \leq b$. Then, we equivalently have $j \in M(\mathbf{s})$ if and only if $s_j \geq \min\left\{\hat{b}, s_i\right\}$, i.e., $c_j(\mathbf{s}_{-j}) = \min\left\{\hat{b}, s_i\right\}$. By Lemma 2($ii$), for all $s_j \geq 0$ and $s_i > c_i(\check{\mathbf{s}}_{-i}) = c_i(\hat{\mathbf{s}}_{-i})$,

$$\begin{aligned} f(\mathbf{s}) &= (s_j - c_j(\mathbf{s}_{-j}))^+ + f(\hat{\mathbf{s}}) \\ &= \left(s_j - \min\left\{\hat{b}, s_i\right\}\right)^+ + f(\hat{\mathbf{s}}). \end{aligned}$$

Therefore, for all $s_j \geq 0$ and $s_i \geq 0$,

$$f(\mathbf{s}) = \left(s_j - \max\left\{\min\left\{\hat{b}, s_i\right\}, c_i(\hat{\mathbf{s}}_{-i})\right\}\right)^+ + f(\hat{\mathbf{s}}). \tag{A14}$$

Case 3. $c_i(\hat{\mathbf{s}}_{-i}) = c_i(\check{\mathbf{s}}_{-i}) \geq b = \bar{s}_{[m^*(\bar{\mathbf{s}})-1]}$. In this case, we have $f(\mathbf{s}) = (s_j - c_i(\hat{\mathbf{s}}_{-i}))^+ + f(\hat{\mathbf{s}})$ for all $s_j \geq 0$ and $s_i \leq c_i(\check{\mathbf{s}}_{-i}) = c_i(\hat{\mathbf{s}}_{-i})$ by applying the same argument in Subcase 1 of Case 1. If $s_i > c_i(\check{\mathbf{s}}_{-i}) = c_i(\hat{\mathbf{s}}_{-i})$, we show that $c_j(\mathbf{s}_{-j}) = c_i(\hat{\mathbf{s}}_{-i})$.

We first show that $c_j(\mathbf{s}_{-j}) \leq c_i(\hat{\mathbf{s}}_{-i})$. Note that $\bar{s}_{[m^*(\bar{\mathbf{s}})-1]} \leq c_i(\hat{\mathbf{s}}_{-i}) = \bar{s}_j$. Then, by the construction of $\bar{\mathbf{s}}$ and $\check{\mathbf{s}}$, $\check{s}_{[m^*(\check{\mathbf{s}})-1]} \leq c_i(\hat{\mathbf{s}}_{-i}) = \check{s}_j$ when $\check{s}_i = c_i(\check{\mathbf{s}}_{-i})$. Because $\check{s}_j$ is not in the last order of the highest $m^*(\check{\mathbf{s}})$ scores when $\check{s}_i = c_i(\check{\mathbf{s}}_{-i})$ and $m^*(\check{\mathbf{s}})$ is unchanged for all $\check{s}_i \geq c_i(\check{\mathbf{s}}_{-i})$ by Lemma 2(i), we must have $j \in M(\check{\mathbf{s}})$ for all $\check{s}_i > c_i(\check{\mathbf{s}}_{-i})$. Therefore, applying Lemma 2(i) again, $c_j(\mathbf{s}_{-j}) \leq \check{s}_j = c_i(\hat{\mathbf{s}}_{-i})$ for any $s_i = \check{s}_i > c_i(\check{\mathbf{s}}_{-i}) = c_i(\hat{\mathbf{s}}_{-i})$.

Next, we prove that $c_j(\mathbf{s}_{-j}) \geq c_i(\hat{\mathbf{s}}_{-i})$. By Equations (A11) and (A12), $f(\hat{\mathbf{s}}) = f(\check{\mathbf{s}}) = s_i - c_i(\hat{\mathbf{s}}_{-i}) + f(\mathbf{s}')$. This implies that $f(\mathbf{s}) = s_i - c_i(\hat{\mathbf{s}}_{-i}) + f(\mathbf{s}')$ for all $s_j \leq c_i(\hat{\mathbf{s}}_{-i})$, and therefore, $c_j(\mathbf{s}_{-j}) \geq c_i(\hat{\mathbf{s}}_{-i})$ by Lemma 2(ii).

Hence, we have $c_j(\mathbf{s}_{-j}) = c_i(\hat{\mathbf{s}}_{-i})$. Applying Lemma 2($ii$) again, for all $s_j \geq 0$ and $s_i > c_i(\check{\mathbf{s}}_{-i}) = c_i(\hat{\mathbf{s}}_{-i})$,

$$\begin{aligned} f(\mathbf{s}) &= (s_j - c_j(\mathbf{s}_{-j}))^+ + f(\hat{\mathbf{s}}) \\ &= (s_j - c_i(\hat{\mathbf{s}}_{-i}))^+ + f(\hat{\mathbf{s}}). \end{aligned}$$

Therefore, we have $f(\mathbf{s}) = (s_j - c_i(\hat{\mathbf{s}}_{-i}))^+ + f(\hat{\mathbf{s}})$ for all $s_j \geq 0$ and $s_i \geq 0$, which also satisfies Equation (A14) by letting $\hat{b} = c_i(\hat{\mathbf{s}}_{-i})$.

Combining Cases 1, 2, and 3, for any $s_j \geq 0$ and $s_i \geq 0$, we have

$$f(\mathbf{s}) = \begin{cases} \left(s_j - \max\left\{\min\left\{\hat{b}, s_i\right\}, c_i(\hat{\mathbf{s}}_{-i})\right\}\right)^+ + f(\hat{\mathbf{s}}) & \text{if } c_i(\hat{\mathbf{s}}_{-i}) = c_i(\check{\mathbf{s}}_{-i}), \\ (s_j - (c_i(\hat{\mathbf{s}}_{-i}) - \min\{c_i(\hat{\mathbf{s}}_{-i}), s_i\} + \min\{c_i(\check{\mathbf{s}}_{-i}), s_i\}))^+ + f(\hat{\mathbf{s}}) & \text{if } c_i(\hat{\mathbf{s}}_{-i}) > c_i(\check{\mathbf{s}}_{-i}). \end{cases}$$

(ii) We consider the following two cases.

Case 1. If $c_i(\hat{\mathbf{s}}_{-i}) = c_i(\check{\mathbf{s}}_{-i})$, then $c_j(\mathbf{s}_{-j}) = \max\left\{\min\left\{\hat{b}, s_i\right\}, c_i(\hat{\mathbf{s}}_{-i})\right\}$, and by symmetry, $c_i(\mathbf{s}_{-i}) = \max\left\{\min\left\{\hat{b}, s_j\right\}, c_i(\hat{\mathbf{s}}_{-i})\right\}$. If $s_i \leq c_i(\hat{\mathbf{s}}_{-i})$, then $s_j \leq s_i \leq c_i(\hat{\mathbf{s}}_{-i}) \leq \hat{b}$. This implies that $c_i(\mathbf{s}_{-i}) = c_j(\mathbf{s}_{-j}) = c_i(\hat{\mathbf{s}}_{-i})$. Therefore, $s_i - c_i(\mathbf{s}_{-i}) = s_i - c_i(\hat{\mathbf{s}}_{-i}) \geq s_j - c_j(\mathbf{s}_{-j})$, which further implies that $f(\mathbf{s} + \delta\mathbf{e}_i) - f(\mathbf{s}) \geq f(\mathbf{s} + \delta\mathbf{e}_j) - f(\mathbf{s})$. If $s_i > c_i(\hat{\mathbf{s}}_{-i})$, then $s_i - c_i(\mathbf{s}_{-i}) = s_i - \max\left\{\min\left\{\hat{b}, s_j\right\}, c_i(\hat{\mathbf{s}}_{-i})\right\} \geq 0$. This implies that $f(\mathbf{s} + \delta\mathbf{e}_i) - f(\mathbf{s}) = \delta \geq f(\mathbf{s} + \delta\mathbf{e}_j) - f(\mathbf{s})$. Hence, $f(\mathbf{s} + \delta\mathbf{e}_i) \geq f(\mathbf{s} + \delta\mathbf{e}_j)$.

Case 2. If $c_i(\hat{\mathbf{s}}_{-i}) > c_i(\check{\mathbf{s}}_{-i})$, then $c_j(\mathbf{s}_{-j}) = c_i(\hat{\mathbf{s}}_{-i}) - \min\{c_i(\hat{\mathbf{s}}_{-i}), s_i\} + \min\{c_i(\check{\mathbf{s}}_{-i}), s_i\}$, and by symmetry, $c_i(\mathbf{s}_{-i}) = c_i(\hat{\mathbf{s}}_{-i}) - \min\{c_i(\hat{\mathbf{s}}_{-i}), s_j\} + \min\{c_i(\check{\mathbf{s}}_{-i}), s_j\}$. If $s_i \leq c_i(\hat{\mathbf{s}}_{-i})$, then $s_i - c_i(\mathbf{s}_{-i}) = s_i - c_i(\hat{\mathbf{s}}_{-i}) + s_j - \min\{c_i(\check{\mathbf{s}}_{-i}), s_j\} \geq s_j - c_i(\hat{\mathbf{s}}_{-i}) + s_i - \min\{c_i(\check{\mathbf{s}}_{-i}), s_i\} = s_j - c_j(\mathbf{s}_{-j})$. This implies that $f(\mathbf{s} + \delta\mathbf{e}_i) - f(\mathbf{s}) \geq f(\mathbf{s} + \delta\mathbf{e}_j) - f(\mathbf{s})$. If $s_i > c_i(\hat{\mathbf{s}}_{-i})$, then $s_i - c_i(\mathbf{s}_{-i}) = s_i - c_i(\hat{\mathbf{s}}_{-i}) + \min\{c_i(\hat{\mathbf{s}}_{-i}), s_j\} - \min\{c_i(\check{\mathbf{s}}_{-i}), s_j\} > \min\{c_i(\hat{\mathbf{s}}_{-i}), s_j\} - \min\{c_i(\check{\mathbf{s}}_{-i}), s_j\} \geq 0$. This implies that $f(\mathbf{s} + \delta\mathbf{e}_i) - f(\mathbf{s}) = \delta \geq f(\mathbf{s} + \delta\mathbf{e}_j) - f(\mathbf{s})$. Hence, $f(\mathbf{s} + \delta\mathbf{e}_i) \geq f(\mathbf{s} + \delta\mathbf{e}_j)$. ■

Proof of Theorem 2: For notational convenience, we suppress q_t, $(s_k^t)_{k\in\{1,2,\ldots,n_t\}\setminus\{i,j\}}$ and the time index for all notations. Let $c_k(\mathbf{s}_{-k}) = c_k^t(q_t, \mathbf{s}_{-k}^t)$, $L_j(\mathbf{s}_{-k}) = L_j^t(q_t, \mathbf{s}_{-j}^t)$, and $U_j(\mathbf{s}_{-k}) = U_j^t(q_t, \mathbf{s}_{-j}^t)$ for $k = 1, 2, \ldots, n_t$. Define $f(s_j, s_i)$ and $g(s_j, s_i)$ similarly as in Equations (A6) and (A7), respectively. For sufficiently small $\delta > 0$, define $\Delta_{s_i} f(s_j, s_i) = f(s_j, s_i + \delta) - f(s_j, s_i)$ and $\Delta_{s_i} g(s_j, s_i) = g(s_j, s_i + \delta) - g(s_j, s_i)$. Then, $\Delta_{s_i} f(s_j, s_i) \in \{0, \delta\}$ by Lemma 2(ii) and $0 \leq \Delta_{s_i} g(s_j, s_i) \leq (1-p)\delta$ by Lemma 1(i) and (iii). Denote $\tilde{\mathbf{s}} = (s_1, \ldots, s_{i-1}, s_i + \delta, s_{i+1}, \ldots, s_{n_t})$. Recall that there are two cases when we define $(L_j(\mathbf{s}_{-j}), U_j(\mathbf{s}_{-j}))$ in the proof of Theorem 1:

(a) If $g(c_j(\mathbf{s}_{-j}), s_i) \leq f(c_j(\mathbf{s}_{-j}), s_i)$, then $L_j(\mathbf{s}_{-j}) = U_j(\mathbf{s}_{-j}) = c_j(\mathbf{s}_{-j})$ (see Equation (A8)).

(b) If $g(c_j(\mathbf{s}_{-j}), s_i) > f(c_j(\mathbf{s}_{-j}), s_i)$, then $U_j(\mathbf{s}_{-j}) = \bar{s}_j$ with $\bar{s}_j > c_j(\mathbf{s}_{-j})$ satisfying $g(\bar{s}_j, s_i) = f(\bar{s}_j, s_i)$, $g(s_j, s_i) > f(s_j, s_i)$ for all $s_j \in (c_j(\mathbf{s}_{-j}), \bar{s}_j)$ and $g(s_j, s_i) < f(s_j, s_i)$ for all $s_j \in (\bar{s}_j, \infty)$; and $L_j(\mathbf{s}_{-j}) = \inf\{s_j \in [0, c_j(\mathbf{s}_{-j})] : g(s_j, s_i) > f(s_j, s_i)\}$ if $g(0, s_i) \leq f(0, s_i)$ and $L_j(\mathbf{s}_{-j}) = -\infty$ if $g(0, s_i) > f(0, s_i)$ (see Equations (A9) and (A10)).

Step 1. We first show that

$$\lim_{s_i\to\infty} L_j(\mathbf{s}_{-j}) = \lim_{s_i\to\infty} U_j(\mathbf{s}_{-j}) = \beta.$$

For any given s_j and sufficiently large s_i, because $\Delta_{s_i} f(s_j, s_i) = \delta > (1-p)\delta \geq \Delta_{s_i} g(s_j, s_i)$, we have $f(s_j, s_i) > g(s_j, s_i)$. This implies that $L_j(\mathbf{s}_{-j}) = U_j(\mathbf{s}_{-j}) = c_j(\mathbf{s}_{-j})$ when s_i is sufficiently large. Because $\lim_{s_i\to\infty} c_j(\mathbf{s}_{-j}) = \beta$ by Lemma 3(i), the result follows.

Step 2. We prove (i) and (ii) together via the following two cases. First note that because $c_j(\mathbf{s}_{-j})$ is continuous in s_i by Lemma 3(i) and both $f(s_j, s_i)$ and $g(s_j, s_i)$ are continuous in (s_j, s_i), it is obvious that $L_j(\mathbf{s}_{-j})$ and $U_j(\mathbf{s}_{-j})$ are also continuous in s_i according to their definitions in (a) and (b).

Case 1. If $c_i(\hat{\mathbf{s}}_{-i}) > c_i(\check{\mathbf{s}}_{-i})$, by Lemma 3($i$), we have

$$f(s_j, s_i) = (s_j - (c_i(\hat{\mathbf{s}}_{-i}) - \min\{c_i(\hat{\mathbf{s}}_{-i}), s_i\} + \min\{c_i(\check{\mathbf{s}}_{-i}), s_i\}))^+ + (s_i - c_i(\hat{\mathbf{s}}_{-i}))^+ + h,$$

where h is some function independent of s_j and s_i. By Lemma 3(i), $c_i(\hat{\mathbf{s}}_{-i}) \geq c_i(\check{\mathbf{s}}_{-i})$. We then consider the following three subcases.

Subcase 1. If $s_i < c_i(\check{\mathbf{s}}_{-i})$, then for any $s_j \geq 0$,

$$f(s_j, s_i) = (s_j - c_i(\hat{\mathbf{s}}_{-i}))^+ + h.$$

We have $\delta > \Delta_{s_i} g(s_j, s_i) \geq 0 = \Delta_{s_i} f(s_j, s_i)$ for all $s_j \geq 0$. In addition, $c_j(\mathbf{s}_{-j}) = c_j(\tilde{\mathbf{s}}_{-j}) = c_i(\hat{\mathbf{s}}_{-i})$. We consider the above two cases (a) and (b) in defining $(L_j(\mathbf{s}_{-j}), U_j(\mathbf{s}_{-j}))$ (see Figure A2(a)):

(1) If $g(c_j(\mathbf{s}_{-j}), s_i) < f(c_j(\mathbf{s}_{-j}), s_i)$, then $g(c_j(\tilde{\mathbf{s}}_{-j}), s_i+\delta) < f(c_j(\tilde{\mathbf{s}}_{-j}), s_i+\delta)$, as δ is sufficiently small. Thus, $L_j(\mathbf{s}_{-j}) = U_j(\mathbf{s}_{-j}) = c_i(\hat{\mathbf{s}}_{-i}) = L_j(\tilde{\mathbf{s}}_{-j}) = U_j(\tilde{\mathbf{s}}_{-j})$.

(2) If $g(c_j(\mathbf{s}_{-j}), s_i) > f(c_j(\mathbf{s}_{-j}), s_i)$, because $\Delta_{s_i} g(s_j, s_i) \geq \Delta_{s_i} f(s_j, s_i) = 0$, $g(\bar{s}_j, s_i + \delta) \geq g(\bar{s}_j, s_i) = f(\bar{s}_j, s_i) = f(\bar{s}_j, s_i + \delta)$. This implies that $U_j(\mathbf{s}_{-j}) = \bar{s}_j \leq U_j(\tilde{\mathbf{s}}_{-j})$. Note that

$$\{s_j \in [0, c_j(\mathbf{s}_{-j})] : g(s_j, s_i) > f(s_j, s_i)\} \subset \{s_j \in [0, c_j(\tilde{\mathbf{s}}_{-j})] : g(s_j, s_i + \delta) > f(s_j, s_i + \delta)\}.$$

This implies that $L_j(\mathbf{s}_{-j}) \geq L_j(\tilde{\mathbf{s}}_{-j})$.

By (1), (2), and the continuity of $L_j(\mathbf{s}_{-j})$ and $U_j(\mathbf{s}_{-j})$ in s_i, $L_j(\mathbf{s}_{-j})$ is decreasing and $U_j(\mathbf{s}_{-j})$ is increasing on $[0, c_i(\check{\mathbf{s}}_{-i}))$.

Subcase 2. If $c_i(\check{\mathbf{s}}_{-i}) \leq s_i < c_i(\hat{\mathbf{s}}_{-i})$, then for any $s_j \geq 0$,

$$f(s_j, s_i) = (s_j - (c_i(\hat{\mathbf{s}}_{-i}) - s_i + c_i(\check{\mathbf{s}}_{-i})))^+ + h.$$

We have $0 \leq \Delta_{s_i} g(s_j, s_i) < \delta = \Delta_{s_i} f(s_j, s_i)$ for all $s_j \geq c_j(\mathbf{s}_{-j})$ and $\delta > \Delta_{s_i} g(s_j, s_i) \geq 0 = \Delta_{s_i} f(s_j, s_i)$ for all $s_j \leq c_j(\tilde{\mathbf{s}}_{-j})$. In addition, $c_j(\mathbf{s}_{-j}) \geq c_j(\tilde{\mathbf{s}}_{-j})$. We also consider the above two cases (a) and (b) (see Figure A2(b)):

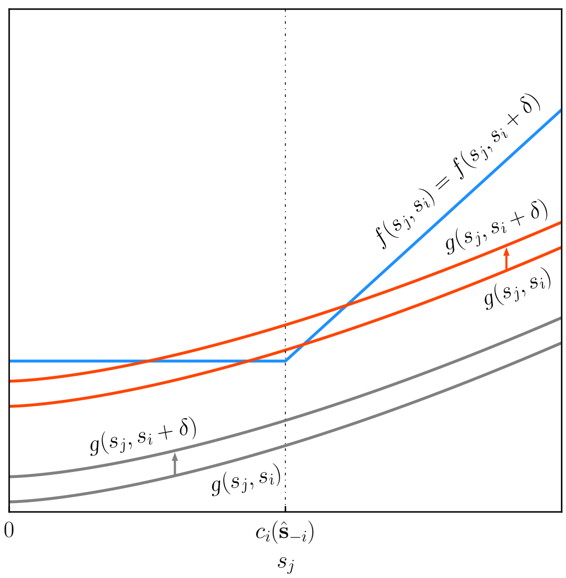
(1) If $g(c_j(\mathbf{s}_{-j}), s_i) < f(c_j(\mathbf{s}_{-j}), s_i)$, then clearly, $U_j(\mathbf{s}_{-j}) = L_j(\mathbf{s}_{-j}) = c_j(\mathbf{s}_{-j}) \geq c_j(\tilde{\mathbf{s}}_{-j}) = U_j(\tilde{\mathbf{s}}_{-j}) = L_j(\tilde{\mathbf{s}}_{-j})$.

(2) If $g(c_j(\mathbf{s}_{-j}), s_i) > f(c_j(\mathbf{s}_{-j}), s_i)$, because $\Delta_{s_i} g(s_j, s_i) < \Delta_{s_i} f(s_j, s_i)$ for all $s_j \geq c_j(\mathbf{s}_{-j})$, $g(\bar{s}_j, s_i + \delta) < f(\bar{s}_j, s_i + \delta)$. This implies that $U_j(\mathbf{s}_{-j}) = \bar{s}_j > U_j(\tilde{\mathbf{s}}_{-j})$. If $L_j(\mathbf{s}_{-j}) \geq c_j(\tilde{\mathbf{s}}_{-j})$, then $L_j(\mathbf{s}_{-j}) \geq c_j(\tilde{\mathbf{s}}_{-j}) \geq L_j(\tilde{\mathbf{s}}_{-j})$. If $L_j(\mathbf{s}_{-j}) \leq c_j(\tilde{\mathbf{s}}_{-j})$, because $\Delta_{s_i} g(s_j, s_i) \geq \Delta_{s_i} f(s_j, s_i)$ for all $s_j \leq c_j(\tilde{\mathbf{s}}_{-j})$, we have

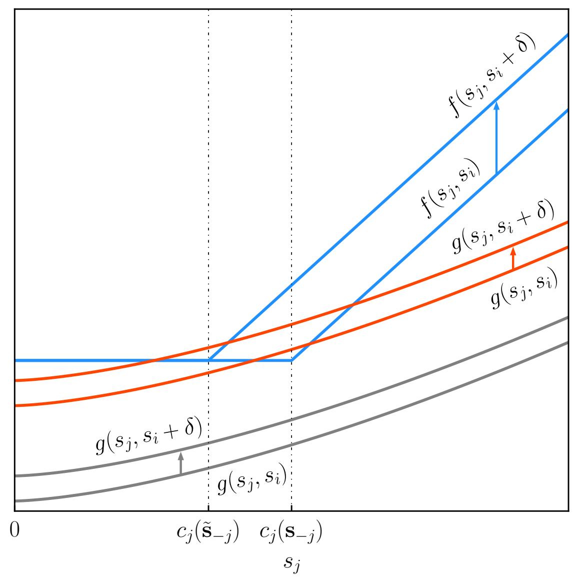
$$\{s_j \in [0, c_j(\tilde{\mathbf{s}}_{-j})] : g(s_j, s_i) > f(s_j, s_i)\} \subset \{s_j \in [0, c_j(\tilde{\mathbf{s}}_{-j})] : g(s_j, s_i + \delta) > f(s_j, s_i + \delta)\}.$$

Figure A2: (Color online) Graphical Illustrations for Cases 1 and 2

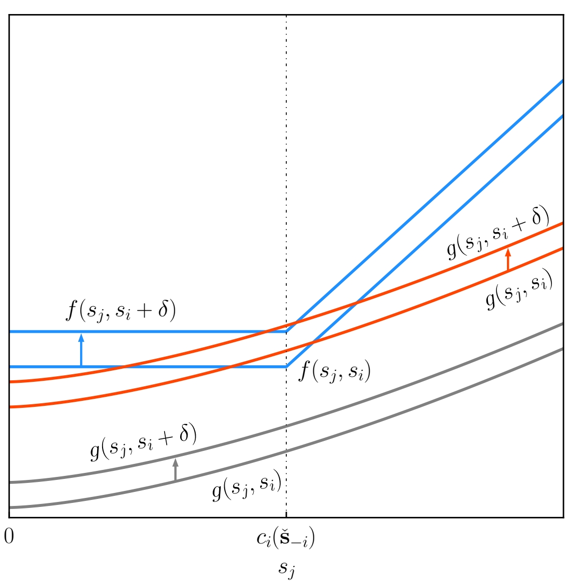
(a) Subcase 1, Case 1



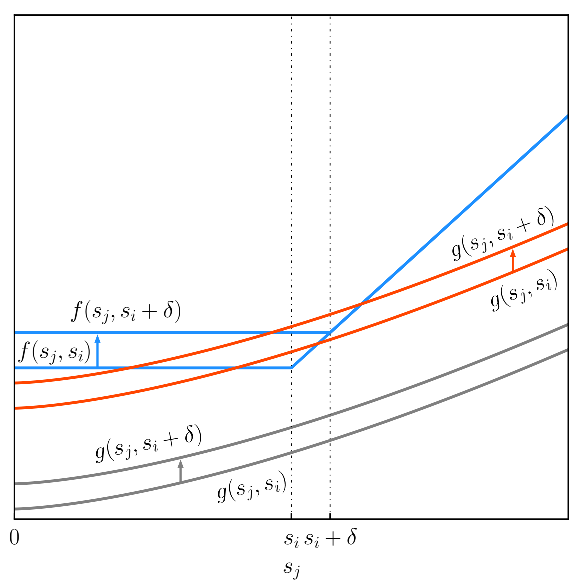
(b) Subcase 2, Case 1



(c) Subcase 3, Case 1



(d) Case 2



This implies that $L_j(\mathbf{s}_{-j}) \geq L_j(\tilde{\mathbf{s}}_{-j})$.

By (1), (2), and the continuity of $L_j(\mathbf{s}_{-j})$ and $U_j(\mathbf{s}_{-j})$ in s_i, $L_j(\mathbf{s}_{-j})$ and $U_j(\mathbf{s}_{-j})$ are decreasing on $[c_i(\check{\mathbf{s}}_{-i}), c_i(\hat{\mathbf{s}}_{-i}))$.

Subcase 3. If $s_i \geq c_i(\hat{\mathbf{s}}_{-i})$, then for any $s_j \geq 0$,

$$f(s_j, s_i) = (s_j - c_i(\check{\mathbf{s}}_{-i}))^+ + s_i - c_i(\hat{\mathbf{s}}_{-i}) + h.$$

We have $0 \leq \Delta_{s_i} g(s_j, s_i) < \delta = \Delta_{s_i} f(s_j, s_i)$ for all $s_j \geq 0$. In addition, $c_j(\mathbf{s}_{-j}) = c_j(\tilde{\mathbf{s}}_{-j}) = c_i(\check{\mathbf{s}}_{-i})$. We consider the above two cases (a) and (b) again (see Figure A2(c)):

(1) If $g(c_j(\mathbf{s}_{-j}), s_i) < f(c_j(\mathbf{s}_{-j}), s_i)$, then clearly, $L_j(\mathbf{s}_{-j}) = U_j(\mathbf{s}_{-j}) = c_i(\check{\mathbf{s}}_{-i}) = L_j(\tilde{\mathbf{s}}_{-j}) = U_j(\tilde{\mathbf{s}}_{-j})$.

(2) If $g(c_j(\mathbf{s}_{-j}), s_i) > f(c_j(\mathbf{s}_{-j}), s_i)$, because $\Delta_{s_i} g(s_j, s_i) < \Delta_{s_i} f(s_j, s_i)$ for all $s_j \geq 0$, $g(\bar{s}_j, s_i + \delta) < f(\bar{s}_j, s_i + \delta)$. This implies that $U_j(\mathbf{s}_{-j}) = \bar{s}_j > U_j(\tilde{\mathbf{s}}_{-j})$. Note that

$$\{s_j \in [0, c_j(\tilde{\mathbf{s}}_{-j})] : g(s_j, s_i + \delta) > f(s_j, s_i + \delta)\} \subset \{s_j \in [0, c_j(\mathbf{s}_{-j})] : g(s_j, s_i) > f(s_j, s_i)\}.$$

This implies that $L_j(\mathbf{s}_{-j}) \leq L_j(\tilde{\mathbf{s}}_{-j})$.

By (1), (2), and the continuity of $L_j(\mathbf{s}_{-j})$ and $U_j(\mathbf{s}_{-j})$ in s_i, $L_j(\mathbf{s}_{-j})$ is increasing and $U_j(\mathbf{s}_{-j})$ is decreasing on $[c_i(\hat{\mathbf{s}}_{-i}), \infty)$.

Combining Subcases 1, 2, and 3, $L_j(\mathbf{s}_{-j})$ is decreasing for $s_i \leq c_i(\hat{\mathbf{s}}_{-i})$ and increasing for $s_i \geq c_i(\hat{\mathbf{s}}_{-i})$; $U_j(\mathbf{s}_{-j})$ is increasing for $s_i \leq c_i(\check{\mathbf{s}}_{-i})$ and decreasing for $s_i \geq c_i(\check{\mathbf{s}}_{-i})$.

Case 2. If $c_i(\hat{\mathbf{s}}_{-i}) = c_i(\check{\mathbf{s}}_{-i})$, by Lemma 3($i$), we have

$$f(s_j, s_i) = \left(s_j - \max\left\{\min\left\{\hat{b}, s_i\right\}, c_i(\hat{\mathbf{s}}_{-i})\right\}\right)^+ + (s_i - c_i(\hat{\mathbf{s}}_{-i}))^+ + h.$$

Recall that $c_i(\hat{\mathbf{s}}_{-i}) = c_i(\check{\mathbf{s}}_{-i}) \leq \hat{b}$. If $s_i < c_i(\check{\mathbf{s}}_{-i}) = c_i(\hat{\mathbf{s}}_{-i})$, $f(s_j, s_i) = (s_j - c_i(\hat{\mathbf{s}}_{-i}))^+ + h$ for all $s_j \geq 0$. Applying the same argument as in Subcase 1 of Case 1, $L_j(\mathbf{s}_{-j})$ is decreasing and $U_j(\mathbf{s}_{-j})$ is increasing on $[0, c_i(\check{\mathbf{s}}_{-i}))$. Similarly, if $s_i > \hat{b}$, $f(s_j, s_i) = \left(s_j - \hat{b}\right)^+ + s_i - c_i(\hat{\mathbf{s}}_{-i}) + h$. Applying the same argument as in Subcase 3 of Case 1 by replacing $c_i(\check{\mathbf{s}}_{-i})$ with $\hat{b}$, $L_j(\mathbf{s}_{-j})$ is increasing and $U_j(\mathbf{s}_{-j})$ is decreasing on $[\hat{b}, \infty)$.

If $c_i(\check{\mathbf{s}}_{-i}) = c_i(\hat{\mathbf{s}}_{-i}) \leq s_i \leq \hat{b}$, then

$$f(s_j, s_i) = (s_j - s_i)^+ + s_i - c_i(\hat{\mathbf{s}}_{-i}) + h.$$

We have $0 \leq \Delta_{s_i} g(s_j, s_i) < \delta = \Delta_{s_i} f(s_j, s_i)$ for all $s_j \leq c_j(\mathbf{s}_{-j}) = s_i$ and $\delta > \Delta_{s_i} g(s_j, s_i) \geq 0 = \Delta_{s_i} f(s_j, s_i)$ for all $s_j \geq c_j(\tilde{\mathbf{s}}_{-j}) = s_i + \delta$. In addition, $c_j(\mathbf{s}_{-j}) \leq c_j(\tilde{\mathbf{s}}_{-j})$. We consider the above two cases (a) and (b) in defining $(L_j(\mathbf{s}_{-j}), U_j(\mathbf{s}_{-j}))$ (see Figure A2(d)):

(1) If $g(c_j(\mathbf{s}_{-j}), s_i) < f(c_j(\mathbf{s}_{-j}), s_i)$, then clearly, $U_j(\mathbf{s}_{-j}) = L_j(\mathbf{s}_{-j}) = c_j(\mathbf{s}_{-j}) \leq c_j(\tilde{\mathbf{s}}_{-j}) = U_j(\tilde{\mathbf{s}}_{-j}) = L_j(\tilde{\mathbf{s}}_{-j})$.

(2) If $g(c_j(\mathbf{s}_{-j}), s_i) > f(c_j(\mathbf{s}_{-j}), s_i)$, because $\Delta_{s_i} g(s_j, s_i) \geq \Delta_{s_i} f(s_j, s_i) = 0$ for all $s_j \geq c_j(\tilde{\mathbf{s}}_{-j})$, $g(\bar{s}_j, s_i+\delta) \geq f(\bar{s}_j, s_i+\delta)$. This implies that $U_j(\mathbf{s}_{-j}) = \bar{s}_j \leq U_j(\tilde{\mathbf{s}}_{-j})$ if $\bar{s}_j \geq c_j(\tilde{\mathbf{s}}_{-j})$. If $\bar{s}_j < c_j(\tilde{\mathbf{s}}_{-j})$, then $U_j(\mathbf{s}_{-j}) = \bar{s}_j \leq c_j(\tilde{\mathbf{s}}_{-j}) \leq U_j(\tilde{\mathbf{s}}_{-j})$. Because $\Delta_{s_i} g(s_j, s_i) < \Delta_{s_i} f(s_j, s_i)$ for all $s_j \leq c_j(\mathbf{s}_{-j})$, $g(L_j(\mathbf{s}_{-j}), s_i + \delta) < f(L_j(\mathbf{s}_{-j}), s_i + \delta)$. This implies that $L_j(\mathbf{s}_{-j}) < L_j(\tilde{\mathbf{s}}_{-j})$.

By (1), (2), and the continuity of $L_j(\mathbf{s}_{-j})$ and $U_j(\mathbf{s}_{-j})$ in s_i, $L_j(\mathbf{s}_{-j})$ and $U_j(\mathbf{s}_{-j})$ are increasing on $[c_i(\hat{\mathbf{s}}_{-i}), \hat{b})$.

Hence, $L_j(\mathbf{s}_{-j})$ is decreasing for $s_i \leq c_i(\hat{\mathbf{s}}_{-i})$ and increasing for $s_i \geq c_i(\hat{\mathbf{s}}_{-i})$; $U_j(\mathbf{s}_{-j})$ is increasing for $s_i \leq \hat{b}$ and decreasing for $s_i \geq \hat{b}$.

Then, (i) and (ii) follow by Cases 1 and 2. ■

For the proof of Theorem 3, we need the following auxiliary lemma related to the optimization problem (3). It shows that if changing a subset of scores does not affect their inclusion on the offer list, then it also does not affect the inclusion of other scores. Let $\delta_i(\mathbf{s})$ denote whether s_i is on the offer list. It is equal to one if s_i is on the offer list and zero otherwise. For any $\mathcal{I} \subset \{1, 2, \ldots, n\}$, let $\tilde{\mathbf{s}}$ be defined similarly to $\mathbf{s}$, except that $(s_i)_{i\in\mathcal{I}}$ is replaced with another vector $(\tilde{s}_i)_{i\in\mathcal{I}}$.

Lemma A1. *If $\delta_i(\tilde{\mathbf{s}}) = \delta_i(\mathbf{s})$ for all $i \in \mathcal{I}$, then $\delta_j(\tilde{\mathbf{s}}) = \delta_j(\mathbf{s})$ for all $j \notin \mathcal{I}$.*

Proof of Lemma A1: The proof is by contradiction. Suppose that there exists some $j \notin \mathcal{I}$ such that $|\delta_j(\mathbf{s}) - \delta_j(\tilde{\mathbf{s}})| = 1$. Let $\mathcal{J}$ denote the set of all these j's. We first compare the optimal value $f(\tilde{\mathbf{s}})$ with the value when keeping $\delta_j(\tilde{\mathbf{s}}) = \delta_j(\mathbf{s})$ for all $j \in \mathcal{J}$:

$$
\begin{aligned}
&f(\tilde{\mathbf{s}}) - \sum_{i\in\mathcal{I}} \tilde{s}_i\delta_i(\tilde{\mathbf{s}}) - \sum_{j\in\mathcal{J}} s_j\delta_j(\mathbf{s}) - \sum_{k\in\{1,2,\ldots,n\}\setminus(\mathcal{I}\bigcup\mathcal{J})} s_k\delta_k(\mathbf{s}) - g\left(\sum_{j\in\mathcal{J}}\delta_j(\mathbf{s}) + \sum_{k\in\{1,2,\ldots,n\}\setminus\mathcal{J}}\delta_k(\mathbf{s})\right)\\
=&\sum_{j\in\mathcal{J}} s_j(\delta_j(\tilde{\mathbf{s}}) - \delta_j(\mathbf{s})) + g\left(m^*(\mathbf{s}) + \sum_{j\in\mathcal{J}}(\delta_j(\tilde{\mathbf{s}}) - \delta_j(\mathbf{s}))\right) - g(m^*(\mathbf{s}))\\
\geq&0, \qquad\qquad (A15)
\end{aligned}
$$

where, by the optimality, the inequality is strict if and only if $\sum_{j\in\mathcal{J}}(\delta_j(\tilde{\mathbf{s}}) - \delta_j(\mathbf{s})) < 0$. When the state is $\mathbf{s}$, we also have

$$\begin{aligned}
&f(\mathbf{s}) - \sum_{j\in\mathcal{J}} s_j\delta_j(\tilde{\mathbf{s}}) - \sum_{k\in\{1,2,\ldots,n\}\setminus\mathcal{J}} s_k\delta_k(\mathbf{s}) - g\left(\sum_{j\in\mathcal{J}}\delta_j(\tilde{\mathbf{s}}) + \sum_{k\in\{1,2,\ldots,n\}\setminus\mathcal{J}}\delta_k(\mathbf{s})\right)\\
=&\sum_{j\in\mathcal{J}} s_j(\delta_j(\mathbf{s})-\delta_j(\tilde{\mathbf{s}})) + g(m^*(\mathbf{s})) - g\left(m^*(\mathbf{s}) + \sum_{j\in\mathcal{J}}(\delta_j(\tilde{\mathbf{s}})-\delta_j(\mathbf{s}))\right)\\
\geq& 0, \qquad\qquad (A16)
\end{aligned}$$

where the inequality is strict if and only if $\sum_{j\in\mathcal{J}}(\delta_j(\tilde{\mathbf{s}})-\delta_j(\mathbf{s})) > 0$.

If $\sum_{j\in\mathcal{J}}(\delta_j(\tilde{\mathbf{s}})-\delta_j(\mathbf{s})) \neq 0$, it is clear that (A15) and (A16) contradict with each other. If $\sum_{j\in\mathcal{J}}(\delta_j(\tilde{\mathbf{s}})-\delta_j(\mathbf{s})) = 0$, because $|\delta_j(\mathbf{s})-\delta_j(\tilde{\mathbf{s}})| = 1$, $|\mathcal{J}|$ is even and $|\mathcal{J}|/2$ of j's in $\mathcal{J}$ satisfy $\delta_j(\mathbf{s}) = 0$ and $\delta_j(\tilde{\mathbf{s}}) = 1$, while the other half i's satisfy $\delta_i(\mathbf{s}) = 1$ and $\delta_i(\tilde{\mathbf{s}}) = 0$. Denote the former set as $\mathcal{J}_1$ and the latter as $\mathcal{J}_2$. Because s_j is on the offer list for all $j \in \mathcal{J}_1$ when the state is $\mathbf{s}$, while s_i is not for all $i \in \mathcal{J}_2$, we must have $\min\{s_j : j \in \mathcal{J}_1\} \geq \max\{s_i : i \in \mathcal{J}_2\}$. In addition, by (A15) and (A16), $\sum_{j\in\mathcal{J}} s_j(\delta_j(\tilde{\mathbf{s}})-\delta_j(\mathbf{s})) = 0 = \sum_{j\in\mathcal{J}} s_j(\delta_j(\mathbf{s})-\delta_j(\tilde{\mathbf{s}})) = \sum_{j\in\mathcal{J}_1} s_j - \sum_{i\in\mathcal{J}_2} s_i$. Therefore, we conclude that $s_j = s_i$ for any $i, j \in \mathcal{J}$, and the optimal policies do not change. ■

Proof of Theorem 3: The proof is by induction. When $n = 1$, by Theorem 1, we let $\mathcal{P}^t_\emptyset = [0, L^t_1(q_t, \mathbf{s}^t_{-1})]$, $\mathcal{P}^t_{\{1\}} = [U^t_1(q_t, \mathbf{s}^t_{-1}), \infty)$ and $\mathcal{C}^t = (L^t_1(q_t, \mathbf{s}^t_{-1}), U^t_1(q_t, \mathbf{s}^t_{-1}))$, and the result holds. Suppose that the result holds for $n = k-1$. For $n = k$, let q_t and $(s^t_i)_{i\in\{k+1,k+2,\ldots,n_t\}}$ be given. For any $s^t_k \in [0,\infty)$, there exists a unique collection of sets $\left\{Q^t_{\mathcal{I}}(s^t_k)\right\}_{\mathcal{I}\subset\{1,2,\ldots,k-1\}}$ in $\mathcal{R}^{k-1}_+$ that satisfies all the statements in Theorem 3. Let $\mathcal{D}^t(s^t_k) = \mathbb{R}^{k-1}_+ \setminus \bigcup_{\mathcal{I}\subset\{1,2,\ldots,k-1\}} \mathcal{Q}^t_{\mathcal{I}}(s^t_k)$. For any $\mathcal{I} \subset \{1,2,\ldots,k-1\}$, we define the following sets:

$$\begin{aligned}
\mathcal{A}^t_{\mathcal{I}}(s^t_k) &= \left\{(s^t_i)_{i\in\{1,2,\ldots,k\}} : \delta^t_k = 0,\ (s^t_i)_{i\in\{1,2,\ldots,k-1\}} \in Q^t_{\mathcal{I}}(s^t_k)\right\},\\
\mathcal{B}^t_{\mathcal{I}}(s^t_k) &= \left\{(s^t_i)_{i\in\{1,2,\ldots,k\}} : \delta^t_k = 1,\ (s^t_i)_{i\in\{1,2,\ldots,k-1\}} \in Q^t_{\mathcal{I}}(s^t_k)\right\},\\
\mathcal{C}^t(s^t_k) &= \left\{(s^t_i)_{i\in\{1,2,\ldots,k\}} : (s^t_i)_{i\in\{1,2,\ldots,k-1\}} \in D^t(s^t_k)\right\}.
\end{aligned}$$

For any $\mathcal{I} \subset \{1,2,\ldots,k\}$, we further define the following sets:

$$\mathcal{P}^t_{\mathcal{I}} = \begin{cases} \bigcup_{s^t_k\in[0,\infty)} \mathcal{A}^t_{\mathcal{I}}(s^t_k) & \text{if } k \notin \mathcal{I},\\ \bigcup_{s^t_k\in[0,\infty)} \mathcal{B}^t_{\mathcal{I}\setminus\{k\}}(s^t_k) & \text{otherwise,} \end{cases}$$

and

$$\mathcal{C}^t = \bigcup_{s^t_k\in[0,\infty)} \mathcal{C}^t(s^t_k).$$

We verify the statements in Theorem 3 one-by-one.

(1) For any nonempty $\mathcal{P}^t_{\mathcal{I}}$ and $\mathcal{P}^t_{\mathcal{J}}$ with distinct $\mathcal{I}, \mathcal{J} \subset \{1, 2, \dots, k\}$, take any point $(s^t_i)_{i\in\{1,2,\dots,k\}} \in \mathcal{P}^t_{\mathcal{I}}$. By the definitions of $\mathcal{A}^t_\cdot(s^t_k)$ and $\mathcal{B}^t_\cdot(s^t_k)$, $\delta^t_i = 1$ for all $i \in \mathcal{I}$ and $\delta^t_j = 0$ for all $j \in \{1, 2, \dots, k\} \setminus \mathcal{I}$. This implies that $(s^t_i)_{i\in\{1,2,\dots,k\}} \notin \mathcal{P}^t_{\mathcal{J}}$. Therefore, $\mathcal{P}^t_{\mathcal{I}} \bigcap \mathcal{P}^t_{\mathcal{J}} = \emptyset$.

(2) If $(s^t_i)_{i\in\{1,2,\dots,k\}} \notin \bigcup_{\mathcal{I}\subset\{1,2,\dots,k\}} \mathcal{P}^t_{\mathcal{I}}$, then $(s^t_i)_{i\in\{1,2,\dots,k\}} \notin \bigcup_{\mathcal{I}\subset\{1,2,\dots,k-1\}} \left(\mathcal{A}^t_{\mathcal{I}}(s^t_k) \bigcup \mathcal{B}^t_{\mathcal{I}}(s^t_k)\right)$ and so $(s^t_i)_{i\in\{1,2,\dots,k-1\}} \notin \bigcup_{\mathcal{I}\subset\{1,2,\dots,k-1\}} Q^t_{\mathcal{I}}(s^t_k)$. This implies that $(s^t_i)_{i\in\{1,2,\dots,k-1\}} \in \mathcal{D}^t(s^t_k)$ and so $(s^t_i)_{i\in\{1,2,\dots,k\}} \in \mathcal{C}^t$. The converse is similar. Thus, $\mathcal{C}^t$ is the complement of $\bigcup_{\mathcal{I}\subset\{1,2,\dots,k\}} \mathcal{P}^t_{\mathcal{I}}$.

(3) By the definitions of $\mathcal{P}^t_{\mathcal{I}}$ and $\mathcal{C}^t$, and by (1) and (2), it is clear that all the nonempty sets from $\mathcal{P}^t_{\mathcal{I}}$ and $\mathcal{C}^t$ form the unique partition of $\mathbb{R}^k_+$ that satisfies (i) and the first statement in (ii). The second statement in (ii) follows from Lemma A1. In addition, that any ray $\left\{(s^t_i)_{i\in\{1,2,\dots,k\}} : s^t_j \geq 0\right\}$ in $\mathbb{R}^k_+$ can sequentially intersect at most three sets $\mathcal{P}^t_{\mathcal{I}}$, $\mathcal{C}^t$ and $\mathcal{P}^t_{\mathcal{J}}$ follows from Theorem 1.

(4) To show the connectedness, we first consider $\mathcal{A}^t_{\mathcal{I}}(s^t_k)$ that is a correspondence from a nonempty set $A \subset [0, \infty)$ into $\mathbb{R}^k_+$. We show by contradiction that A is connected, or equivalently, A is an interval. Suppose that A is not an interval. Then, there exist two points $\tilde{s}^t_k < s^t_k$ such that $\tilde{s}^t_k \notin A$ and $s^t_k \in A$. Take any point $(s^t_i)_{i\in\{1,2,\dots,k\}} \in \mathcal{A}^t_{\mathcal{I}}(s^t_k)$. Because $\delta^t_i = 0$, by (ii), lowering s^t_i dose not change any δ^t_j, $j = 1, 2, \dots, n_t$. Using (ii) again, we have $((s^t_i)_{i\in\{1,2,\dots,k\}}, \tilde{s}^t_k) \in \mathcal{A}^t_{\mathcal{I}}(\tilde{s}^t_k)$, which contradicts to $\tilde{s}^t_k \notin A$. (From this argument we can see that $A = [0, s_A]$ for some $s_A \geq 0$.) By the induction hypothesis, $\mathcal{Q}^t_{\mathcal{I}}(s^t_k)$ is connected, so $\mathcal{A}^t_{\mathcal{I}}(s^t_k)$ is connected. Also, because the thresholds $L^t_i(q_t, \mathbf{s}^t_{-i})$ and $U^t_i(q_t, \mathbf{s}^t_{-i})$, $i = 1, 2, \dots, k-1$, are continuous in s^t_k, we conclude that $\mathcal{P}^t_{\mathcal{I}} = \mathcal{A}^t_{\mathcal{I}}(A)$ is connected. Similar arguments can be applied to $P^t_{\mathcal{I}}$ with $k \in \mathcal{I}$.

(5) For any $\mathcal{I} \subset \{1, 2, \dots, k\}$ with $|\mathcal{I}| = m \in \{0, 1, \dots, k\}$, take any point $(s^t_i)_{i\in\{1,2,\dots,k\}} \in \mathcal{P}^t_{\mathcal{I}}$. If $k \notin \mathcal{I}$, then $(s^t_i)_{i\in\{1,2,\dots,k\}} \in \mathcal{A}^t_{\mathcal{I}}(s^t_k)$ and so $(s^t_i)_{i\in\{1,2,\dots,k-1\}} \in \mathcal{Q}^t_{\mathcal{I}}(s^t_k)$. By the induction hypothesis, $m^*_t(q_t, \mathbf{s}^t) = \mathcal{M}^{k-1}_t(m; s^t_k)$ for some increasing mapping $\mathcal{M}^{k-1}_t : \{0, 1, \dots, k-1\} \to \{0, 1, \dots, n_t\}$ (by (iii)). If $k \in \mathcal{I}$, then $(s^t_i)_{i\in\{1,2,\dots,k\}} \in \mathcal{B}^t_{\mathcal{I}\setminus\{k\}}(s^t_k)$ and so $(s^t_i)_{i\in\{1,2,\dots,k-1\}} \in \mathcal{Q}^t_{\mathcal{I}\setminus\{k\}}(s^t_k)$. Similarly, $m^*_t(q_t, \mathbf{s}^t) = \mathcal{M}^{k-1}_t(m-1; s^t_k)$. Define

$$\mathcal{M}_t(m; s^t_k) = \begin{cases} \mathcal{M}^{k-1}_t(m; s^t_k) & \text{if } k \notin \mathcal{I}, \\ \mathcal{M}^{k-1}_t(m-1; s^t_k) & \text{otherwise,} \end{cases}$$

which is s^t_k-dependent. We next show that $\mathcal{M}_t(m; s^t_k)$ is constant on $\mathcal{P}^t_{\mathcal{I}}$ so s^t_k can be removed. Take any two $(s^t_i)_{i\in\{1,2,\dots,k\}}, (\tilde{s}^t_i)_{i\in\{1,2,\dots,k\}} \in \mathcal{P}^t_{\mathcal{I}}$. Let $\tilde{\mathbf{s}}^t$ be defined the same as $\mathbf{s}^t$ except that $(s^t_i)_{i\in\{1,2,\dots,k\}}$ is replaced with $(\tilde{s}^t_i)_{i\in\{1,2,\dots,k\}}$. If $k \notin \mathcal{I}$, by (ii), δ^t_i does not change for $i = 1, 2, \dots, n_t$. This implies that $\mathcal{M}^{k-1}_t(m; s^t_k) = m^*_t(q_t, \mathbf{s}^t) = m^*_t(q_t, \tilde{\mathbf{s}}^t) = \mathcal{M}^{k-1}_t(m; \tilde{s}^t_k)$. The same result holds for $k \in \mathcal{I}$. So we let $\mathcal{M}_t(m) = \mathcal{M}_t(m; s^t_k)$ on $\mathcal{P}^t_{\mathcal{I}}$.

We shall show that $\mathcal{M}_t(m)$ is increasing in m. For any $\mathcal{I}, \mathcal{J} \subset \{1, 2, \dots, k\}$ with $|\mathcal{I}| \geq |\mathcal{J}|$, there are four cases: (a) if $k \notin \mathcal{I} \bigcup \mathcal{J}$, then $\mathcal{M}_t(|\mathcal{I}|) = \mathcal{M}_t^{k-1}(|\mathcal{I}|) \geq \mathcal{M}_t^{k-1}(|\mathcal{J}|) = \mathcal{M}_t(|\mathcal{J}|)$; (b) if $k \in \mathcal{I} \bigcap \mathcal{J}$, then $\mathcal{M}_t(|\mathcal{I}|) = \mathcal{M}_t^{k-1}(|\mathcal{I}|-1) \geq \mathcal{M}_t^{k-1}(|\mathcal{J}|-1) = \mathcal{M}_t(|\mathcal{J}|)$; (c) if $k \in \mathcal{J} \setminus \mathcal{I}$, then $\mathcal{M}_t(|\mathcal{I}|) = \mathcal{M}_t^{k-1}(|\mathcal{I}|) \geq \mathcal{M}_t^{k-1}(|\mathcal{J}|-1) = \mathcal{M}_t(|\mathcal{J}|)$; (d) if $k \in \mathcal{I} \setminus \mathcal{J}$, then $\mathcal{M}_t(|\mathcal{I}|) = \mathcal{M}_t^{k-1}(|\mathcal{I}|-1)$ and $\mathcal{M}_t(|\mathcal{J}|) = \mathcal{M}_t^{k-1}(|\mathcal{J}|)$. For $|\mathcal{I}|-1 \geq |\mathcal{J}|$, we still have $\mathcal{M}_t(|\mathcal{I}|) \geq \mathcal{M}_t(|\mathcal{J}|)$. For $|\mathcal{I}|-1 < |\mathcal{J}|$, or equivalently, $|\mathcal{I}| = |\mathcal{J}|$, there must exist an $l \in \{1, 2, \ldots, k-1\}$ such that $l \notin \mathcal{I}$. By interchanging values in the lth and kth coordinates of the score vector $(s_i^t)_{i \in \{1,2,\ldots,k\}} \in \mathcal{P}_{\mathcal{I}}^t$, we have $\delta_i^t = 1$ for all $i \in \mathcal{I} \bigcup \{l\} \setminus \{k\}$ and $\delta_j^t = 0$ for all $j \in \mathcal{I} \bigcup \{l\} \setminus \{k\}$, so the new vector is in $\mathcal{P}_{\mathcal{I} \bigcup \{l\} \setminus \{k\}}^t$. Therefore, $\mathcal{M}_t(|\mathcal{I}|) = \mathcal{M}_t(|\mathcal{I} \bigcup \{l\} \setminus \{k\}|) = \mathcal{M}_t^{k-1}(|\mathcal{I} \bigcup \{l\} \setminus \{k\}|) = \mathcal{M}_t^{k-1}(|\mathcal{J}|) = \mathcal{M}_t(|\mathcal{J}|)$.

The induction is completed. ■

Proof of Theorem 4: (i) Without loss of generality, suppose that all score states are in descending order, i.e., $\mathbf{s}^t = (s_{[1]}^t, s_{[2]}^t, \ldots, s_{[n_t]}^t), t = 1, 2, \ldots, T$. For each pair $i, j \in \{1, 2, \ldots, n_t\}$ with $i < j$, we first show that $V_t(q_t, \mathbf{s}^t + \delta \mathbf{e}_i) \geq V_t(q_t, \mathbf{s}^t + \delta \mathbf{e}_j)$ by induction. The result obviously holds for $T+1$. Suppose $V_{t+1}(q_{t+1}, \mathbf{s}^{t+1} + \delta \mathbf{e}_i) \geq V_{t+1}(q_{t+1}, \mathbf{s}^{t+1} + \delta \mathbf{e}_j)$ for each pair $i, j \in \{1, 2, \ldots, n_{t+1}\}$ with $i < j$. For notational brevity, denote $\mathbf{s}' = \mathbf{s}^t + \delta \mathbf{e}_i$ and $\mathbf{s}'' = \mathbf{s}^t + \delta \mathbf{e}_j$, suppress q_t, and define

$$f(\mathbf{s}^t) = \max_{1 \leq m_t \leq n_t} \left\{ \sum_{i=1}^{m_t} s_{[i]}^t + \mathbb{E} V_{t+1}(q_t + m_t, \mathbf{S}^{t+1}(\mathbf{s}^t, 1)) \right\}.$$

Then, by Lemma 3(ii), $f(\mathbf{s}') \geq f(\mathbf{s}'')$. We shall show that

$$\mathbb{E} V_{t+1}(q_t, \mathbf{S}^{t+1}(\mathbf{s}', 0)) - \mathbb{E} V_{t+1}(q_t, \mathbf{S}^{t+1}(\mathbf{s}'', 0)) \geq 0. \tag{A17}$$

Expanding the left side of Equation (A17) yields

$$\begin{aligned}
&\mathbb{E} V_{t+1}(q_t, \mathbf{S}^{t+1}(\mathbf{s}', 0)) - \mathbb{E} V_{t+1}(q_t, \mathbf{S}^{t+1}(\mathbf{s}'', 0)) \\
=&(1-p)^2 \left(\mathbb{E}\left[V_{t+1}(q_t, \mathbf{S}^{t+1}(\mathbf{s}', 0)) \middle| W_i^t = 0, W_j^t = 0 \right] - \mathbb{E}\left[V_{t+1}(q_t, \mathbf{S}^{t+1}(\mathbf{s}'', 0)) \middle| W_i^t = 0, W_j^t = 0 \right] \right) \\
&+ p(1-p) \left(\mathbb{E}\left[V_{t+1}(q_t, \mathbf{S}^{t+1}(\mathbf{s}', 0)) \middle| W_i^t = 0, W_j^t = 1 \right] - \mathbb{E}\left[V_{t+1}(q_t, \mathbf{S}^{t+1}(\mathbf{s}'', 0)) \middle| W_i^t = 0, W_j^t = 1 \right] \right) \\
&- p(1-p) \left(\mathbb{E}\left[V_{t+1}(q_t, \mathbf{S}^{t+1}(\mathbf{s}'', 0)) \middle| W_i^t = 1, W_j^t = 0 \right] - \mathbb{E}\left[V_{t+1}(q_t, \mathbf{S}^{t+1}(\mathbf{s}', 0)) \middle| W_i^t = 1, W_j^t = 0 \right] \right) \\
&+ p^2 \left(\mathbb{E}\left[V_{t+1}(q_t, \mathbf{S}^{t+1}(\mathbf{s}', 0)) \middle| W_i^t = 1, W_j^t = 1 \right] - \mathbb{E}\left[V_{t+1}(q_t, \mathbf{S}^{t+1}(\mathbf{s}'', 0)) \middle| W_i^t = 1, W_j^t = 1 \right] \right) \\
\geq& 0,
\end{aligned}$$

where the inequality follows because the first term on the left side of the inequality is positive by the induction hypothesis, the sum of the second and third terms is positive by Lemma 1(i), and the last term is simply zero. Hence, we have $V_t(q_t, \mathbf{s}^t + \delta \mathbf{e}_i) \geq V_t(q_t, \mathbf{s}^t + \delta \mathbf{e}_j)$. The induction is complete. We then obtain

$$
\begin{aligned}
V_t(q_t, \mathbf{s}^t) =& V_t(q_t, \mathbf{s}^t + \delta \mathbf{e}_i - \delta \mathbf{e}_i) \\
\geq& V_t(q_t, \mathbf{s}^t + \delta \mathbf{e}_j - \delta \mathbf{e}_i).
\end{aligned}
$$

Next, we prove (ii) and (iii) together. Denote $\tilde{\mathbf{s}}^t = \mathbf{s}^t + \delta \mathbf{e}_j - \delta \mathbf{e}_i$. By Equation (A17),

$$
\begin{aligned}
\mathbb{E} V_{t+1}(q_t, \mathbf{S}^{t+1}(\mathbf{s}^t, 0)) =& \mathbb{E} V_{t+1}(q_t, \mathbf{S}^{t+1}(\mathbf{s}^t + \delta \mathbf{e}_i - \delta \mathbf{e}_i, 0)) \\
\geq& \mathbb{E} V_{t+1}(q_t, \mathbf{S}^{t+1}(\mathbf{s}^t + \delta \mathbf{e}_j - \delta \mathbf{e}_i, 0)) \\
=& \mathbb{E} V_{t+1}(q_t, \mathbf{S}^{t+1}(\tilde{\mathbf{s}}^t, 0)).
\end{aligned}
$$

Thus, it suffices to show that $f(\mathbf{s}^t) = f(\tilde{\mathbf{s}}^t)$ for both $i, j \in M_t(q_t, \mathbf{s}^t)$ and $i, j \notin M_t(q_t, \mathbf{s}^t)$. To this end, define $(c_j^t(q_t, \mathbf{s}_{-j}^t), \hat{\mathbf{s}}^t, \check{\mathbf{s}}^t, \bar{\mathbf{s}}^t, b_t, \hat{b}_t, \beta)$ similarly as in Theorem 2. By Lemma 3(i), we have

$$
c_j^t(q_t, \mathbf{s}_{-j}^t) = \begin{cases} \max\left\{\min\left\{\hat{b}_t, s_i^t\right\}, c_i^t(q_t, \hat{\mathbf{s}}_{-i}^t)\right\} & \text{if } c_i^t(q_t, \hat{\mathbf{s}}_{-i}^t) = c_i^t(q_t, \check{\mathbf{s}}_{-i}^t), \\ c_i^t(q_t, \hat{\mathbf{s}}_{-i}^t) - \min\left\{c_i^t(q_t, \hat{\mathbf{s}}_{-i}^t), s_i^t\right\} + \min\left\{c_i^t(q_t, \check{\mathbf{s}}_{-i}^t), s_i^t\right\} & \text{if } c_i^t(q_t, \hat{\mathbf{s}}_{-i}^t) > c_i^t(q_t, \check{\mathbf{s}}_{-i}^t). \end{cases}
$$

By symmetry, $c_i^t(q_t, \mathbf{s}_{-i}^t)$ is defined similarly. By Lemma 2(i), $i, j \in M_t(q_t, \mathbf{s}^t)$ is equivalent to $s_i^t \geq c_i^t(q_t, \mathbf{s}_{-i}^t)$ and $s_j^t \geq c_j^t(q_t, \mathbf{s}_{-j}^t)$; and $i, j \notin M_t(q_t, \mathbf{s}^t)$ is equivalent to $s_i^t < c_i^t(q_t, \mathbf{s}_{-i}^t)$ and $s_j^t < c_j^t(q_t, \mathbf{s}_{-j}^t)$. We then consider the following two cases.

Case 1. If $c_i^t(q_t, \check{\mathbf{s}}_{-i}^t) = c_i^t(q_t, \hat{\mathbf{s}}_{-i}^t)$, then $s_i^t \geq c_i^t(q_t, \mathbf{s}_{-i}^t)$ and $s_j^t \geq c_j^t(q_t, \mathbf{s}_{-j}^t)$ imply that $s_i^t > s_j^t \geq \hat{b}_t$. By Lemma 3$(i)$,

$$
\begin{aligned}
f(\mathbf{s}^t) =& \left(s_j^t - \max\left\{\min\left\{\hat{b}_t, s_i^t\right\}, c_i^t(q_t, \hat{\mathbf{s}}_{-i}^t)\right\}\right)^+ + f(\hat{\mathbf{s}}^t) \\
=& s_j^t - \hat{b}_t + s_i^t - c_i^t(q_t, \hat{\mathbf{s}}_{-i}^t) + h,
\end{aligned}
$$

where h is some function independent of s_j^t and s_i^t. Because $s_i^t - \delta \geq s_j^t + \delta \geq \hat{b}_t$, $f(\tilde{\mathbf{s}}^t) = (s_j^t - \delta) - \hat{b}_t + (s_i^t + \delta) - c_i^t(q_t, \hat{\mathbf{s}}_{-i}^t) + h = f(\mathbf{s}^t)$.

Similarly, $s_i^t < c_i^t(q_t, \mathbf{s}_{-i}^t)$ and $s_j^t < c_j^t(q_t, \mathbf{s}_{-j}^t)$ imply that $s_j^t < s_i^t < c_i^t(q_t, \hat{\mathbf{s}}_{-i}^t)$. By Lemma 3$(i)$,

$$
\begin{aligned}
f(\mathbf{s}^t) =& \left(s_j^t - \max\left\{\min\left\{\hat{b}_t, s_i^t\right\}, c_i^t(q_t, \hat{\mathbf{s}}_{-i}^t)\right\}\right)^+ + f(\hat{\mathbf{s}}^t) \\
=& h,
\end{aligned}
$$

which also implies that $f(\mathbf{s}^t) = f(\tilde{\mathbf{s}}^t)$.

Case 2. If $c_i^t(q_t, \hat{\mathbf{s}}_{-i}^t) > c_i^t(q_t, \check{\mathbf{s}}_{-i}^t)$, then $s_i^t \geq c_i^t(q_t, \mathbf{s}_{-i}^t)$ and $s_j^t \geq c_j^t(q_t, \mathbf{s}_{-j}^t)$ imply that $s_j^t \geq c_i^t(q_t, \check{\mathbf{s}}_{-i}^t)$ if $s_i^t > c_i^t(q_t, \hat{\mathbf{s}}_{-i}^t)$ or $s_j^t \geq c_i^t(q_t, \check{\mathbf{s}}_{-i}^t) + c_i^t(q_t, \hat{\mathbf{s}}_{-i}^t) - s_i^t$ if $\frac{1}{2}(c_i^t(q_t, \check{\mathbf{s}}_{-i}^t) + c_i^t(q_t, \hat{\mathbf{s}}_{-i}^t)) < s_i^t \leq$

$c_i^t(q_t, \hat{\mathbf{s}}_{-i}^t)$. By Lemma 3($i$), if $s_i^t > c_i^t(q_t, \hat{\mathbf{s}}_{-i}^t)$,

$$\begin{aligned} f(\mathbf{s}^t) &= \left(s_j^t - \left(c_i^t(q_t, \hat{\mathbf{s}}_{-i}^t) - \min\left\{c_i^t(q_t, \hat{\mathbf{s}}_{-i}^t), s_i^t\right\} + \min\left\{c_i^t(q_t, \check{\mathbf{s}}_{-i}^t), s_i^t\right\}\right)\right)^+ + f(\hat{\mathbf{s}}^t) \\ &= s_j^t - c_i^t(q_t, \check{\mathbf{s}}_{-i}^t) + s_i^t - c_i^t(q_t, \hat{\mathbf{s}}_{-i}^t) + h. \end{aligned}$$

If $\frac{1}{2}(c_i^t(q_t, \check{\mathbf{s}}_{-i}^t) + c_i^t(q_t, \hat{\mathbf{s}}_{-i}^t)) \le s_i^t < c_i^t(q_t, \hat{\mathbf{s}}_{-i}^t)$, we also have $f(\mathbf{s}^t) = s_j^t - c_i^t(q_t, \check{\mathbf{s}}_{-i}^t) + s_i^t - c_i^t(q_t, \hat{\mathbf{s}}_{-i}^t) + h$. Therefore, $f(\mathbf{s}^t) = f(\tilde{\mathbf{s}}^t)$.

Similarly, $s_i^t < c_i^t(q_t, \mathbf{s}_{-i}^t)$ and $s_j^t < c_j^t(q_t, \mathbf{s}_{-j}^t)$ imply that $s_j^t < s_i^t \le \frac{1}{2}(c_i^t(q_t, \check{\mathbf{s}}_{-i}^t) + c_i^t(q_t, \hat{\mathbf{s}}_{-i}^t))$ or $s_j^t < c_i^t(q_t, \check{\mathbf{s}}_{-i}^t) + c_i^t(q_t, \hat{\mathbf{s}}_{-i}^t) - s_i^t$ if $\frac{1}{2}(c_i^t(q_t, \check{\mathbf{s}}_{-i}^t) + c_i^t(q_t, \hat{\mathbf{s}}_{-i}^t)) < s_i^t \le c_i^t(q_t, \hat{\mathbf{s}}_{-i}^t)$. By Lemma 3($i$), if $s_j^t < s_i^t \le \frac{1}{2}(c_i^t(q_t, \check{\mathbf{s}}_{-i}^t) + c_i^t(q_t, \hat{\mathbf{s}}_{-i}^t))$,

$$\begin{aligned} f(\mathbf{s}^t) &= \left(s_j^t - \left(c_i^t(q_t, \hat{\mathbf{s}}_{-i}^t) - \min\left\{c_i^t(q_t, \hat{\mathbf{s}}_{-i}^t), s_i^t\right\} + \min\left\{c_i^t(q_t, \check{\mathbf{s}}_{-i}^t), s_i^t\right\}\right)\right)^+ + f(\hat{\mathbf{s}}^t) \\ &= \left(s_j^t - \left(c_i^t(q_t, \hat{\mathbf{s}}_{-i}^t) - s_i^t + \min\left\{c_i^t(q_t, \check{\mathbf{s}}_{-i}^t), s_i^t\right\}\right)\right)^+ + h \\ &= h. \end{aligned}$$

If $\frac{1}{2}c_i^t(q_t, \check{\mathbf{s}}_{-i}^t) + c_i^t(q_t, \hat{\mathbf{s}}_{-i}^t) < s_i^t \le c_i^t(q_t, \hat{\mathbf{s}}_{-i}^t)$, we also have $f(\mathbf{s}^t) = h$. Hence, $f(\mathbf{s}^t) = f(\tilde{\mathbf{s}}^t)$.

Then, (ii) and (iii) follow by Cases 1 and 2. ■

To facilitate the derivation of the proof of Theorem 5, we provide some lemmas.

Lemma A2. *For any $l \in \{1, 2, \ldots, k+1\}$, the following statements hold.*

(i) $\hat{V}_t(q_t, \mathbf{y}^t)$ is convex increasing in $\mathbf{y}^{t,l}$.

(ii) $\nabla_{y_j^{t,l}} \hat{V}_t(q_t, \mathbf{y}^t) \le 1$, $j = 1, 2, \ldots, n_t^l$.

(iii) $\nabla_{y_j^{t,l}} \mathbb{E}\hat{V}_{t+1}(q_t + \sum_{i=1}^{k+1} m_t^i, \mathbf{Y}^{t+1}(\mathbf{y}^t, \mathbf{m}_t, a_t)) \le 1 - p$, $j = 1, 2, \ldots, n_t^l$.

Proof of Lemma A2: (i) The proof is by induction. $\hat{V}_{T+1}(q_{T+1}, \mathbf{y}^{T+1})$ is obviously convex increasing in $\mathbf{y}^{T+1,l}$ for any $l \in \{1, 2, \ldots, k+1\}$. Suppose $\hat{V}_{t+1}(q_{t+1}, \mathbf{y}^{t+1})$ is convex increasing in $\mathbf{y}^{t+1,l}$. By the same argument showing that $\mathbb{E}V_{t+1}(q_t, \mathbf{S}^{t+1}(\mathbf{s}^t, 0))$ is convex increasing in $\mathbf{s}^t$ in the proof of Lemma 1(i), $\mathbb{E}\hat{V}_{t+1}(q_t + \sum_{i=1}^{k+1} m_t^i, \mathbf{Y}^{t+1}(\mathbf{y}^t, \mathbf{m}_t, a_t))$ is convex increasing in $\mathbf{y}^{t,l}$.

Next, we show that $\hat{J}_t(q_t, \mathbf{m}_t, \mathbf{y}^t)$ is convex increasing in $\mathbf{y}^{t,l}$. This is obvious because both $\sum_{i=1}^{m_t^l} y_{[i]}^{t,l}$ and $\mathbb{E}\hat{V}_{t+1}\left(q_t + \sum_{i=1}^{k+1} m_t^i, \mathbf{Y}^{t+1}(\mathbf{y}^t, \mathbf{m}_t, 1)\right)$ are convex increasing in $\mathbf{y}^{t,l}$, and the sum of two convex increasing functions is still convex increasing. Note that $\hat{V}_t(q_t, \mathbf{y}^t)$ can be written as

$$\hat{V}_t(q_t, \mathbf{y}^t) = \max\left\{\mathbb{E}\hat{V}_{t+1}(q_t, \mathbf{Y}^{t+1}(\mathbf{y}^t, \mathbf{0}, 0)), \hat{J}_t(q_t, \mathbf{0}, \mathbf{y}^t), \hat{J}_t(q_t, (1, 0, \ldots, 0), \mathbf{y}^t), \ldots, \hat{J}_t(q_t, \mathbf{n}_t, \mathbf{y}^t)\right\},$$

which is the maximum of $\prod_{i=1}^{k+1}(n_t^i + 1) + 1$ convex increasing functions of $\mathbf{y}^{t,l}$. Therefore, $\hat{V}_t(q_t, \mathbf{y}^t)$ is convex increasing in $\mathbf{y}^{t,l}$.

(*ii*) and (*iii*) follow by applying the same arguments in the proofs of Lemma 1(*ii*) and (*iii*), respectively. Hence, we omit the proofs for the sake of brevity. ■

Lemma A3. *Let* $\mathbf{s} = (s_1, s_2, \ldots, s_n) \in \mathbb{R}^n_+$ *and* $s_{[i]}$ *be the* i*th largest value in* $\mathbf{s}$. *Denote* $\mathbf{s}_{[m:n]} = (s_{[m]}, s_{[m+1]}, \ldots, s_{[n]})$ *for* $m \le n$ *and set* $\mathbf{s}_{[n+1:n]} = 0$. *Let*

$$f(\mathbf{s}) = \max_{1\le m\le n} \left\{ \sum_{i=1}^{m} s_{[i]} + g_m(\mathbf{s}_{[m+1:n]}) \right\}, \tag{A18}$$

where g_m *is any real-valued function of* $\mathbf{s}_{[m+1:n]}$ *that satisfies* $\nabla_{s_i} g_m \le 1$ *for all* $i \in \{1, 2, \ldots, n\}$. *Denote* $m^*(\mathbf{s})$ *as the largest maximizer in (A18) and let* $M(\mathbf{s}) = \left\{i \in \{1, 2, \ldots, n\} : s_i \ge s_{[m^*(\mathbf{s})]}\right\}$. *Then, for any* $j \in \{1, 2, \ldots, n\}$, *the set* $\left\{s_j \ge 0 : s_j \ge s_{[m^*(\mathbf{s})]}\right\}$ *is nonempty and its minimum is attainable. Let* $c_j(\mathbf{s}_{-j}) = \min\left\{s_j \ge 0 : s_j \ge s_{[m^*(\mathbf{s})]}\right\}$. *There exists a constant* $\overline{m} \in \{1, 2, \ldots, n\}$ *such that* $m^*(\mathbf{s}) = \overline{m}$ *for all* $s_j \ge c_j(\mathbf{s}_{-j})$. *Moreover,* $j \in M(\mathbf{s})$ *if and only if* $s_j \ge c_j(\mathbf{s}_{-j})$.

Proof of Lemma A3: For notational brevity, let $\tilde{\mathbf{s}} = (s_1, \ldots, s_{j-1}, \tilde{s}_j, s_{j+1}, \ldots, s_n)$ and $\mathbf{s}' = (s_1, \ldots, s_{j-1}, s'_j, s_{j+1}, \ldots, s_n)$. Let $s_j = \max_{i\in\{1,2,\ldots,n\}\setminus\{j\}} s_i$. Because $s_j = s_{[1]} \ge s_{[m^*(\mathbf{s})]}$, the set $\left\{s_j \ge 0 : s_j \ge s_{[m^*(\mathbf{s})]}\right\}$ is nonempty. Define $c_j(\mathbf{s}_{-j}) = \inf\left\{s_j \ge 0 : s_j \ge s_{[m^*(\mathbf{s})]}\right\}$ and let $\mathbf{s}^c = (s_1, \ldots, s_{j-1}, c_j(\mathbf{s}_{-j}), s_{j+1}, \ldots, s_n)$. To show that the infimum can be replaced by the minimum, it suffices to show that $c_j(\mathbf{s}_{-j}) \ge s^c_{[m^*(\mathbf{s}^c)]}$.

Take any $\varepsilon > 0$ and let $\tilde{s}_j = c_j(\mathbf{s}_{-j}) + \varepsilon$. We first show that $j \in M(\tilde{\mathbf{s}})$. By the definition of $c_j(\mathbf{s}_{-j})$, there exists an $s'_j \in \left\{s_j \ge 0 : s_j \ge s_{[m^*(\mathbf{s})]}\right\}$ such that $s'_j < c_j(\mathbf{s}_{-j}) + \varepsilon = \tilde{s}_j$. Then, we have $\tilde{s}_j > s'_j \ge s'_{[m^*(\mathbf{s}')]}$ and by the optimality of $m^*(\mathbf{s}')$, for any given $m \in \{1, 2, \ldots, n\}$,

$$\sum_{i=1}^{m^*(\mathbf{s}')} s'_{[i]} + g_{m^*(\mathbf{s}')}(\mathbf{s}'_{[m^*(\mathbf{s}')+1:n]}) - \sum_{i=1}^{m} s'_{[i]} - g_m(\mathbf{s}'_{[m+1:n]}) \ge 0.$$

Because $\nabla_{s_j} g_m \le 1$, the left side of the above inequality increases with s'_j (by fixing $m^*(\mathbf{s}')$). Then,

$$\begin{aligned}
&\sum_{i=1}^{m^*(\mathbf{s}')} \tilde{s}_{[i]} + g_{m^*(\mathbf{s}')}(\tilde{\mathbf{s}}_{[m^*(\mathbf{s}')+1:n]}) - \sum_{i=1}^{m} \tilde{s}_{[i]} - g_m(\tilde{\mathbf{s}}_{[m+1:n]}) \\
\ge &\sum_{i=1}^{m^*(\mathbf{s}')} s'_{[i]} + g_{m^*(\mathbf{s}')}(\mathbf{s}'_{[m^*(\mathbf{s}')+1:n]}) - \sum_{i=1}^{m} s'_{[i]} - g_m(\mathbf{s}'_{[m+1:n]}) \\
\ge &0.
\end{aligned} \tag{A19}$$

Since this holds for all $m \in \{1, 2, \ldots, n\}$ and by definition, the second inequality is strict for all $m > m^*(\mathbf{s}')$, we have $m^*(\tilde{\mathbf{s}}) = m^*(\mathbf{s}')$ and $j \in M(\tilde{\mathbf{s}})$. Then, we have $\tilde{s}_j \ge \tilde{s}_{[m^*(\tilde{\mathbf{s}})]} \ge s^c_{[m^*(\tilde{\mathbf{s}})]}$, where the second inequality holds because $\tilde{\mathbf{s}}$ only differs from $\mathbf{s}^c$ in the jth component and $\tilde{s}_j > s^c_j$. Because

$m^*(\mathbf{s})$ is the largest maximizer in (A18), it is clear that $m^*(\mathbf{s})$ is a step function of s_j and is right continuous in s_j. Therefore,

$$c_j(\mathbf{s}_{-j}) = \lim_{\varepsilon \to 0+} (c_j(\mathbf{s}_{-j}) + \varepsilon) = \lim_{\varepsilon \to 0+} \tilde{s}_j \geq \lim_{\varepsilon \to 0+} s^c_{[m^*(\tilde{\mathbf{s}})]} = s^c_{[m^*(\mathbf{s}^c)]}.$$

Hence, $c_j(\mathbf{s}_{-j}) \geq s^c_{[m^*(\mathbf{s}^c)]}$, and the minimum of $\left\{s_j \geq 0 : s_j \geq s_{[m^*(\mathbf{s})]}\right\}$ is attainable.

Using the same argument as that for Equation (A19), one can verify that the second statement holds. The third statement follows from the same argument for proving the third statement of Lemma 2(i). ■

Proof of Theorem 5: (i) The maximal reward when the recruiter stops can be written as

$$\begin{aligned}\max_{\mathbf{0} \leq \mathbf{m}_t \leq \mathbf{n}_t} \hat{J}_t(q_t, \mathbf{m}_t, \mathbf{y}^t) &= \max_{0 \leq m^1_t \leq n^1_t} \max_{\substack{0 \leq m^l_t \leq n^l_t, \\ 2 \leq l \leq k+1}} \hat{J}_t(q_t, \mathbf{m}^t, \mathbf{y}^t) \\ &= \max_{0 \leq m^1_t \leq n^1_t} \left\{ \sum_{i=1}^{m^1_t} y^{t,1}_{[i]} + g_t(m^1_t) \right\}, \qquad \text{(A20)}\end{aligned}$$

where $g_t(m^1_t) = \max_{0 \leq m^l_t \leq n^l_t, 2 \leq l \leq k+1} \left\{ \sum_{l=2}^{k+1} \sum_{i=1}^{m^l_t} y^{t,l}_{[i]} + \mathbb{E}\hat{V}_{t+1}\left(q_t + \sum_{l=1}^{k+1} m^l_t, \mathbf{Y}^{t+1}(\mathbf{y}^t, \mathbf{m}_t, 1)\right)\right\}$. Note that with slight modifications, all of the results in Lemmas 2 and 3 can be extended to the constraint set starting with 0. Then, the properties of (2) also hold true for (A20), and the proofs of Theorems 1, 2, and 4 can be directly applied to prove the same results for applicants on $\mathbf{y}^{t,1}$. Thus, we omit the proofs for the sake of brevity.

(ii)(1) To prove the first statement, we first consider the maximal reward when the recruiter stops:

$$\max_{\mathbf{0} \leq \mathbf{m}_t \leq \mathbf{n}_t} \hat{J}_t(q_t, \mathbf{m}_t, \mathbf{y}^t) = \max_{0 \leq m^l_t \leq n^l_t} \left\{ \sum_{i=1}^{m^l_t} y^{t,l}_{[i]} + h_t(m^l_t, \mathbf{y}^{t,l}) \right\},$$

where $h_t(m^l_t, \mathbf{y}^{t,l}) = \max_{0 \leq m^{l'}_t \leq n^{l'}_t, l' \neq l} \left\{ \sum_{l' \neq l} \sum_{i=1}^{m^{l'}_t} y^{t,l'}_{[i]} + \mathbb{E}\hat{V}_{t+1}\left(q_t + \sum_{l'=1}^{k+1} m^{l'}_t, \mathbf{Y}^{t+1}(\mathbf{y}^t, \mathbf{m}_t, 1)\right)\right\}$. Let

$$\hat{f}(q_t, \mathbf{y}^{t,l}) = \max_{1 \leq m^l_t \leq n^l_t} \left\{ \sum_{i=1}^{m^l_t} y^{t,l}_{[i]} + h_t(m^l_t, \mathbf{y}^{t,l}) \right\}. \qquad \text{(A21)}$$

Denote $\hat{m}^l_t(q_t, \mathbf{y}^{t,l})$ as the largest maximizer in (A21).

Because $0 \leq \nabla_{y^{t,l}_j} h_t(m^l_t, \mathbf{y}^{t,l}) \leq 1 - p < 1$ (Lemma A2(i) and (iii)), by Lemma A3, we can define $c^{t,l}_j(q_t, \mathbf{y}^{t,l}_{-j}) = \min\left\{y^{t,l}_j \geq 0 : y^{t,l}_j \geq y^{t,l}_{[\hat{m}^l_t(q_t, \mathbf{y}^{t,l})]}\right\}$. It is clear that $\hat{f}(q_t, \mathbf{y}^{t,l})$ is linearly increasing in $y^{t,l}_j$ with slope 1 if $y^{t,l}_j \geq c^{t,l}_j(q_t, \mathbf{y}^{t,l}_{-j})$ and is convex increasing in $y^{t,l}_j$ with slope less than $1 - p$ if

$y_j^{t,l} < c_j^{t,l}(q_t, \mathbf{y}_{-j}^{t,l})$. Because the maximal reward when the recruiter waits $\mathbb{E}\hat{V}_{t+1}(q_t, \mathbf{Y}^{t+1}(\mathbf{y}^t, \mathbf{0}, 0))$ is also convex increasing in $y_j^{t,l}$ with slope less than $1-p$, letting

$$U_j^{t,l} = \inf\left\{y_j^{t,l} \geq c_j^{t,l}(q_t, \mathbf{y}_{-j}^{t,l}) : \hat{f}(q_t, \mathbf{y}^{t,l}) \geq \max\left\{h_t(0, \mathbf{y}^{t,l}), \mathbb{E}\hat{V}_{t+1}(q_t, \mathbf{Y}^{t+1}(\mathbf{y}^t, \mathbf{0}, 0))\right\}\right\},$$

one can verify that the results hold.

$(ii)(2)$ Without loss of generality, suppose that all score states are in descending order, i.e., $y_1^{t,l} \geq y_2^{t,l} \geq \cdots \geq y_{n_t^l}^{t,l}$, $t = 1, 2, \ldots, T$, $l = 1, 2, \ldots, k+1$. For any $\varepsilon > 0$, any $l \in \{1, 2, \ldots, k+1\}$, and any pair $i, j \in \left\{1, 2, \ldots, n_t^l\right\}$ with $i < j$, we first show that $\hat{V}_t(q_t, \mathbf{y}^{t'}) \geq \hat{V}_t(q_t, \mathbf{y}^{t''})$ by induction, where $\mathbf{y}^{t'} = (\mathbf{y}^{t,1}, \ldots, \mathbf{y}^{t,l-1}, \mathbf{y}^{t,l} + \varepsilon\mathbf{e}_i, \mathbf{y}^{t,l+1}, \ldots, \mathbf{y}^{t,k+1})$ and $\mathbf{y}^{t''} = (\mathbf{y}^{t,1}, \ldots, \mathbf{y}^{t,l-1}, \mathbf{y}^{t,l} + \varepsilon\mathbf{e}_j, \mathbf{y}^{t,l+1}, \ldots, \mathbf{y}^{t,k+1})$. The result obviously holds for $T+1$. Suppose $\hat{V}_{t+1}(q_{t+1}, \mathbf{y}^{(t+1)'}) \geq \hat{V}_{t+1}(q_{t+1}, \mathbf{y}^{(t+1)''})$. For any given $m_t^l \in \left\{0, 1, \ldots, n_t^l\right\}$, we first show that $\hat{J}_t(q_t, \mathbf{m}_t, \mathbf{y}^{t'}) \geq \hat{J}_t(q_t, \mathbf{m}_t, \mathbf{y}^{t''})$ through the following two cases.

Case 1. $m_t^l > 0$ and $y_i^{t,l} \geq y_{[m_t^l]}^{t,l}$. For any $y_{k_1}^{t,l} \geq y_{[m_t^l]}^{t,l}$ with $k_1 \in \left\{1, 2, \ldots, n_t^l\right\}$, we have $\nabla_{y_{k_1}^{t,l}} \hat{J}_t(q_t, \mathbf{m}_t, \mathbf{y}^t) = \nabla_{y_{k_1}^{t,l}} \sum_{i=1}^{m_t^l} y_{[i]}^{t,l} = 1$. Note that $\hat{J}_t(q_t, \mathbf{m}_t, \mathbf{y}^t)$ is convex increasing in $\mathbf{y}^{t,l}$. Therefore, $\nabla_{y_{k_1}^{t,l}} \hat{J}_t(q_t, \mathbf{m}_t, \mathbf{y}^t) \leq 1$ for all $y_{k_1}^{t,l} \geq 0$. Then, $\nabla_{y_i^{t,l}} \hat{J}_t(q_t, \mathbf{m}_t, \mathbf{y}^t) = 1 \geq \nabla_{y_j^{t,l}} \hat{J}_t(q_t, \mathbf{m}_t, \mathbf{y}^t)$, which implies that $\hat{J}_t(q_t, \mathbf{m}_t, \mathbf{y}^{t'}) \geq \hat{J}_t(q_t, \mathbf{m}_t, \mathbf{y}^{t''})$.

Case 2. Either $m_t^l = 0$ or $m_t^l > 0$ and $y_i^{t,l} < y_{[m_t^l]}^{t,l}$. For any $y_{k_1}^{t,l}$ and $y_{k_2}^{t,l}$ in $\mathbf{Y}^{t+1,l-1}(\mathbf{y}^t, \mathbf{m}_t, 1)$ with $y_{k_1}^{t,l} \geq y_{k_2}^{t,l}$,

$$\begin{aligned}
\nabla_{y_{k_1}^{t,l}} \hat{J}_t(q_t, \mathbf{m}_t, \mathbf{y}^t) =& \nabla_{y_{k_1}^{t,l}} \mathbb{E}\hat{V}_{t+1}\left(q_t + \sum_{l'=1}^{k+1} m_t^{l'}, \mathbf{Y}^{t+1}(\mathbf{y}^t, \mathbf{m}_t, 1)\right) \\
\geq& \nabla_{y_{k_2}^{t,l}} \mathbb{E}\hat{V}_{t+1}\left(q_t + \sum_{l'=1}^{k+1} m_t^{l'}, \mathbf{Y}^{t+1}(\mathbf{y}^t, \mathbf{m}_t, 1)\right) \\
=& \nabla_{y_{k_2}^{t,l}} \hat{J}_t(q_t, \mathbf{m}_t, \mathbf{y}^t),
\end{aligned}$$

where the inequality can be proved by applying the same argument as that for Theorem 4(i) because the following two required conditions hold: (1) $\hat{V}_{t+1}(\cdot, \mathbf{y}^{t+1})$ is convex increasing in $\mathbf{y}^{t+1,l-1}$ by Lemma A2(i); and (2) $\hat{V}_{t+1}(q_{t+1}, \mathbf{y}^{(t+1)'}) \geq \hat{V}_{t+1}(q_{t+1}, \mathbf{y}^{(t+1)''})$ by the induction hypothesis. Then, $\nabla_{y_i^{t,l}} \hat{J}_t(q_t, \mathbf{m}_t, \mathbf{y}^t) \geq \nabla_{y_j^{t,l}} \hat{J}_t(q_t, \mathbf{m}_t, \mathbf{y}^t)$, which implies that $\hat{J}_t(q_t, \mathbf{m}_t, \mathbf{y}^{t'}) \geq \hat{J}_t(q_t, \mathbf{m}_t, \mathbf{y}^{t''})$.

By Cases 1 and 2, $\hat{J}_t(q_t, \mathbf{m}_t, \mathbf{y}^{t'}) \geq \hat{J}_t(q_t, \mathbf{m}_t, \mathbf{y}^{t''})$ holds for all $\mathbf{0} \leq \mathbf{m}_t \leq \mathbf{n}_t$.

Note that $\mathbb{E}\hat{V}_{t+1}(q_t, \mathbf{Y}^{t+1}(\mathbf{y}^{t'}, \mathbf{0}, 0)) \geq \mathbb{E}\hat{V}_{t+1}(q_t, \mathbf{Y}^{t+1}(\mathbf{y}^{t''}, \mathbf{0}, 0))$ follows by the same argument as that for Theorem 4(i). Hence, $\hat{V}_t(q_t, \mathbf{y}^{t'}) \geq \hat{V}_t(q_t, \mathbf{y}^{t''})$. The induction is complete.

It then follows that

$$\hat{V}_t(q_t, \mathbf{y}^t) = \hat{V}_t(q_t, (\mathbf{y}^{t,1}, \ldots, \mathbf{y}^{t,l-1}, \mathbf{y}^{t,l} + \delta\mathbf{e}_i - \delta\mathbf{e}_i, \mathbf{y}^{t,l+1}, \ldots, \mathbf{y}^{t,k+1}))$$

$$\geq \hat{V}_t(q_t, (\mathbf{y}^{t,1}, \ldots, \mathbf{y}^{t,l-1}, \mathbf{y}^{t,l} + \delta \mathbf{e}_j - \delta \mathbf{e}_i, \mathbf{y}^{t,l+1}, \ldots, \mathbf{y}^{t,k+1})).$$

(iii) We prove this by contradiction. Suppose that applicant i is not hired. The idea is to show that hiring i and not hiring j (called system 2) leads to a higher expected reward, denoted by R_2, than the expected reward generated by hiring j and not hiring i (called system 1), denoted by R_1. Expanding $R_2 - R_1$ yields

$$R_2 - R_1 =$$

$$y_i^{t,l'} - y_j^{t,l} + \mathbb{E}\hat{V}_{t+1}\left(q_t + \sum_{l=1}^{k+1} m_t^{l^*}, \tilde{\mathbf{Y}}^{t+1}(\mathbf{y}^t, \mathbf{m}_t^* - \mathbf{e}_{l'} + \mathbf{e}_l, 1)\right) - \mathbb{E}\hat{V}_{t+1}\left(q_t + \sum_{l=1}^{k+1} m_t^{l^*}, \mathbf{Y}^{t+1}(\mathbf{y}^t, \mathbf{m}_t^*, 1)\right), \tag{A22}$$

where $\mathbf{m}_t^* = (m_t^{1^*}, m_t^{2^*}, \ldots, m_t^{(k+1)^*})$ is the number of offers to make in system 1, and therefore, $\mathbf{m}_t^* - \mathbf{e}_{l'} + \mathbf{e}_l$ is the number of offers to make in system 2. $\tilde{\mathbf{Y}}^{t+1}$ has the same element in each coordinate as $\mathbf{Y}^{t+1}$ except that in the $(l-1)$th coordinate,

$$\tilde{\mathbf{Y}}^{t+1,l-1}(\mathbf{y}^t, \mathbf{m}_t^* - \mathbf{e}_{l'} + \mathbf{e}_l, 1) = \hat{\mathbf{Y}}^{t+1}\left(y_j^{t,l}, y_{[m_t^{l^*}+1]}^{t,l}, y_{[m_t^{l^*}+2]}^{t,l}, \ldots, y_{[n_t^l]}^{t,l}\right),$$

and in the $(l'-1)$th coordinate (if $l' > 1$),

$$\tilde{\mathbf{Y}}^{t+1,l'-1}(\mathbf{y}^t, \mathbf{m}_t^* - \mathbf{e}_{l'} + \mathbf{e}_l, 1) = \hat{\mathbf{Y}}^{t+1}(\mathbf{z}^{t,l'}),$$

where $\mathbf{z}^{t,l'}$ is the score vector containing all of the scores in $\left(y_{[m_t^{(l')^*}+1]}^{t,l'}, y_{[m_t^{(l')^*}+2]}^{t,l'}, \ldots, y_{[n_t^{l'}]}^{t,l}\right)$ except $y_i^{t,l'}$.

If $l' = 1$, then system 2 has one more score $y_t^{t,l'}$ than system 1 after the recruiter extends offers; in this case, a simple sample-path argument can show that system 2 has a higher expected reward starting from period $t+1$ than system 1. Thus, Equation (A22) implies that $R_2 - R_1 \geq y_i^{t,l'} - y_j^{t,l} \geq 0$.

If $l' > 1$, then

$$\begin{aligned}
&\mathbb{E}\hat{V}_{t+1}\left(q_t + \sum_{l=1}^{k+1} m_t^{l^*}, \tilde{\mathbf{Y}}^{t+1}(\mathbf{y}^t, \mathbf{m}_t^* - \mathbf{e}_{l'} + \mathbf{e}_l, 1)\right) - \mathbb{E}\hat{V}_{t+1}\left(q_t + \sum_{l=1}^{k+1} m_t^{l^*}, \mathbf{Y}^{t+1}(\mathbf{y}^t, \mathbf{m}_t^*, 1)\right)\\
\geq &\mathbb{E}\hat{V}_{t+1}\left(q_t + \sum_{l=1}^{k+1} m_t^{l^*}, \tilde{\mathbf{Y}}^{t+1}(\mathbf{y}^t, \mathbf{m}_t^* - \mathbf{e}_{l'} + \mathbf{e}_l, 1)\right) - \mathbb{E}\hat{V}_{t+1}\left(q_t + \sum_{l=1}^{k+1} m_t^{l^*}, \bar{\mathbf{Y}}^{t+1}(\mathbf{y}^t, \mathbf{m}_t^* - \mathbf{e}_{l'} + \mathbf{e}_l, 1)\right)\\
\geq &y_j^{t,l} - y_i^{t,l'},
\end{aligned} \tag{A23}$$

where $\bar{\mathbf{Y}}^{t+1}$ has the same element in each coordinate as $\tilde{\mathbf{Y}}^{t+1}$ except that in the $(l-1)$th coordinate,

$$\bar{\mathbf{Y}}^{t+1,l-1}(\mathbf{y}^t, \mathbf{m}_t^* - \mathbf{e}_{l'} + \mathbf{e}_l, 1) = \hat{\mathbf{Y}}^{t+1}\left(y_i^{t,l'}, y_{[m_t^{l^*}+1]}^{t,l}, y_{[m_t^{l^*}+2]}^{t,l}, \ldots, y_{[n_t^l]}^{t,l}\right).$$

In other words, we move applicant i from the older list, $\mathbf{y}^{t,l'}$, to the more recent one, $\mathbf{y}^{t,l}$; in this case, a simple sample-path argument can show that moving an applicant from an older list to a younger one does not reduce the expected reward. Hence, the first inequality in (A23) holds. The second equality follows by Lemma A2(ii). Therefore, we also have $R_2 - R_1 \geq 0$. ■